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Superstabilization of Bose systems: II. Bose condensations and equivalence of ensembles

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Abstract
In Bru (2002 *J. Phys. A: Math. Gen.* **35** 8969), we present a general method of superstabilization corresponding to the addition of the 'forward scattering' interaction to some non-superstable model. Here we complete the thermodynamic study done in this paper and we prove that the 'standard' thermodynamic behaviour of the non-superstable model is preserved in its superstabilized form in the grand-canonical ensemble. In particular, the Bose condensation phenomena persist in the new Bose gas. Moreover, this method ensures the strong equivalence for the new superstabilized model between the canonical and grand-canonical ensembles in the thermodynamic limit (Georgii H-O 1994 *Probab. Theory Related Fields* **99** 171–95) and then it gives another way to analyse the canonical (infinite volume) Gibbs state of the first non-superstable model.

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1. Introduction

To fix the notation, we recall first some facts about the *imperfect Bose gas* (IBG), see [3–9]. Enclosed in a cubic box $\Lambda = \mathop{\times}\limits_{\alpha=1}^{d} L \subset \mathbb{R}^d$, the IBG [3] is defined by

$$H_\Lambda^{IBG} \equiv T_\Lambda + \frac{\lambda}{V} N_\Lambda^2 \qquad \lambda > 0 \tag{1.1}$$

with

$$T_\Lambda \equiv \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k \tag{1.2}$$

$$N_\Lambda \equiv \sum_{k\in\Lambda^*} a_k^* a_k. \tag{1.3}$$

Here $\varepsilon_k = \hbar^2 k^2/2m$ represents the one-particle energy spectrum and the sums run over the set

$$\Lambda^* = \left\{ k \in \mathbb{R}^d : k_\alpha = \frac{2\pi n_\alpha}{L}, n_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1, 2, \ldots, d \right\}$$

i.e. we consider periodic boundary conditions. The operators $a_k^\# = \{a_k^* \text{ or } a_k\}$ are the usual boson creation/annihilation operators in the one-particle state $\psi_k(x) = V^{-\frac{1}{2}} \mathrm{e}^{\mathrm{i}kx}$, $k \in \Lambda^*$, $x \in \Lambda$, acting on the boson Fock space

$$\mathcal{F}_\Lambda^B \equiv \bigoplus_{n=0}^{+\infty} \mathcal{H}_B^{(n)} \tag{1.4}$$

where

$$\mathcal{H}_B^{(n)} \equiv (L^2(\Lambda^n))_{\mathrm{symm}} \tag{1.5}$$

are the symmetrized n-particle Hilbert spaces appropriate for bosons $\left(\mathcal{H}_B^{(0)} = \mathbb{C}\right)$. In contrast with the *perfect Bose gas* (PBG) (1.2), the IBG (1.1) is superstable [11] and so, given an inverse temperature $\beta > 0$, the corresponding grand-canonical IBG pressure

$$p_\Lambda^{IBG}(\beta,\mu) = \frac{1}{\beta V} \ln \mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^{IBG} - \mu N_\Lambda)}\right) \tag{1.6}$$

exists for any chemical potential $\mu \in \mathbb{R}$ even in the thermodynamic limit:

$$Q^{IBG} \equiv \left\{(\beta > 0, \mu \in \mathbb{R}) : \lim_\Lambda p_\Lambda^{IBG}(\beta,\mu) < +\infty\right\} = Q^S \equiv \{\beta > 0\} \times \{\mu \in \mathbb{R}\}. \tag{1.7}$$

Moreover, for $d \geqslant 3$, there is a critical chemical potential

$$\mu_c^{IBG}(\beta) \equiv 2\lambda \rho^{PBG}(\beta, 0) > 0 \tag{1.8}$$

such that the Bose–Einstein (BE) condensation, in the PBG, *persists* in the IBG (1.1) for $\mu > \mu_c^{IBG}(\beta)$, see [5, 6, 9]. Here

$$\rho^{PBG}(\beta,0) \equiv \sup_{\alpha<0} \rho^{PBG}(\beta,\alpha) = \lim_{\alpha\to 0^-} \rho^{PBG}(\beta,\alpha) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d k \, (\mathrm{e}^{\beta\varepsilon_k} - 1)^{-1} < +\infty \tag{1.9}$$

is the critical density for the PBG in high dimensions $d \geqslant 3$ with

$$\rho^{PBG}(\beta,\alpha) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}^d k \, (\mathrm{e}^{\beta(\varepsilon_k - \alpha)} - 1)^{-1}. \tag{1.10}$$

In particular, the papers [5, 6] show that the conventional BE condensation with density $\rho_0^{IBG}(\beta,\mu)$ is a condensation on the zero-mode,

$$\rho_0^{IBG}(\beta,\mu) \equiv \lim_\Lambda \frac{1}{V} \langle a_0^* a_0 \rangle_{H_\Lambda^{IBG}}(\beta,\mu) = 2\left(\frac{\mu - \mu_c^{IBG}(\beta)}{\lambda}\right) > 0 \qquad \text{for} \quad \mu > \mu_c^{IBG}(\beta) \tag{1.11}$$

where $\langle - \rangle_{H_\Lambda^{IBG}}(\beta,\mu)$ represents the (finite volume) Gibbs state for the Hamiltonian H_Λ^{IBG} (1.1) in the grand-canonical ensemble (β,μ).

In fact, in the *grand-canonical* ensemble but for a *fixed* particle density $\rho > 0$ one has

$$\rho_0^{IBG}(\beta,\rho) = \rho_0^{IBG}(\beta,\mu^{IBG}(\rho)) = \sup\{0, \rho - \rho^{PBG}(\beta,0)\} \tag{1.12}$$

where for any $\rho > 0$, $\mu^{IBG}(\rho)$ is the unique chemical potential solution of equation

$$\rho^{IBG}(\beta,\mu) \equiv \lim_\Lambda \frac{1}{V} \langle N_\Lambda \rangle_{H_\Lambda^{IBG}}(\beta,\mu) = \rho. \tag{1.13}$$

Thus, the conventional BE condensate density $\rho_0^{IBG}(\beta,\rho)$ (1.12) for the IBG (1.1) coincides with that for the PBG (1.2),

$$\rho_0^{PBG}(\beta,\rho) \equiv \lim_{\Lambda} \frac{1}{V}\langle a_0^* a_0\rangle_{T_\Lambda}\left(\beta, \alpha_\Lambda^{PBG}(\rho)\right) = \sup\{0, \rho - \rho^{PBG}(\beta, 0)\} \quad (1.14)$$

i.e.

$$\rho_0^{PBG}(\beta,\rho) = \rho_0^{IBG}(\beta,\rho) \quad (1.15)$$

for any *fixed* particle density $\rho > 0$ in the grand-canonical ensemble. Here $\alpha_\Lambda^{PBG}(\rho)$ in (1.14) is the unique chemical potential verifying

$$\rho_\Lambda^{PBG}\left(\beta, \alpha = \alpha_\Lambda^{PBG}(\rho)\right) \equiv \frac{1}{V}\langle N_\Lambda\rangle_{T_\Lambda}\left(\beta, \alpha_\Lambda^{PBG}(\rho)\right) = \rho \quad (1.16)$$

for any $\rho > 0$, with $\langle - \rangle_{T_\Lambda}(\beta, \alpha)$ representing the finite volume Gibbs state for the Hamiltonian T_Λ (1.2) also in the *grand-canonical* ensemble (β, α).

Then, adding the interaction $(\lambda/V)N_\Lambda^2$ to a *non-superstable* Hamiltonian (but stable [11])

$$H_\Lambda^X \equiv \sum_{k\in\Lambda^*} \varepsilon_k a_k^* a_k + U_\Lambda^X = T_\Lambda + U_\Lambda^X \quad (1.17)$$

does not seem to change the intrinsic thermodynamic properties. In particular, the phenomenon of Bose condensation seems to persist in the new superstabilized model

$$H_\Lambda^{SX} \equiv H_\Lambda^X + \frac{\lambda}{V}N_\Lambda^2 \qquad \lambda > 0. \quad (1.18)$$

Another example of such influence could be seen in [12, 13] where the interaction $(\lambda/V)N_\Lambda^2$ stabilizes this specific Bose system for any $\mu \in \mathbb{R}$ without destroying the coexistence of two Bose condensations [13]: a first one, *non-conventional*, in the zero-mode (type I) which coexists for high fixed densities with a second one, *conventional* and *non-extensive* (type III), see (3.40)–(3.45) below.

In [1], we have presented a general method of superstabilization corresponding to (1.18). Then assuming some *sufficient* conditions on the *non-superstable* Bose system X, we have explicitly found the thermodynamic functions (free-energy density, grand-canonical pressure and particle density) of the new *superstabilized* model SX using the model X. Then this paper proposes to complete this thermodynamic analysis [1] by studying the Bose condensation phenomena. Then, following the work of Georgii [2], by analysing the notions of equivalence of ensembles (canonical/grand-canonical) we explain the main interest of this method of superstabilization: it gives a new way to find the thermodynamic behaviour of the *non-superstable* model in the *canonical* ensemble such as, for example, the existence of Bose condensations.

Hence, in section 2 we briefly recall the thermodynamic results of [1]. Then the section 3 derives the existence of Bose condensations for the superstabilized Hamiltonian assuming some Bose condensations in the first (*non-superstable*) Bose gas X. In section 4, we explain how the canonical ensemble is related to the grand-canonical ensemble on the level of Gibbs states for the superstabilized Bose gas SX and we describe a method to deduce some thermodynamic properties (Bose condensations) for the *non-superstable* Hamiltonian in the canonical ensemble. We reserve section 5 for concluding remarks and discussions. Some reminders about classification of Bose condensations and generating functionals are presented in two appendices.

2. Thermodynamic study [1]

2.1. Set-up of the problem

The Hamiltonians H_Λ^X (1.17) and H_Λ^{SX} (1.18) are well defined on the boson Fock space $\mathcal{F}_\Lambda^B$ (1.4). H_Λ^X verifies

$$\left[H_\Lambda^X, N_\Lambda\right] = 0. \tag{2.1}$$

Then, for a fixed particle density $\rho > 0$, we define by $f_\Lambda^X(\beta,\rho)$ and $f_\Lambda^{SX}(\beta,\rho)$ the free-energy densities,

$$f_\Lambda^X(\beta,\rho) \equiv -\frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{H}_B^{(n)}}\left(\left\{\mathrm{e}^{-\beta H_\Lambda^X}\right\}^{(n)}\right) \qquad f_\Lambda^{SX}(\beta,\rho) \equiv -\frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{H}_B^{(n)}}\left(\left\{\mathrm{e}^{-\beta H_\Lambda^{SX}}\right\}^{(n)}\right) \tag{2.2}$$

where $n = [V\rho]$ denotes the integer part of $V\rho$ and

$$A^{(n)} \equiv A\lceil \mathcal{H}_B^{(n)} \tag{2.3}$$

is the restriction of any operator A acting on the boson Fock space $\mathcal{F}_\Lambda^B$ (1.4) to $\mathcal{H}_B^{(n)}$ (1.5).

Using, in the grand-canonical ensemble, two chemical potentials α and μ, respectively for the models X (1.17) and SX (1.18), the corresponding pressures are defined by

$$p_\Lambda^X(\beta,\alpha) \equiv \frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^X-\alpha N_\Lambda)}\right) \qquad p_\Lambda^{SX}(\beta,\mu) \equiv \frac{1}{\beta V}\ln \mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^{SX}-\mu N_\Lambda)}\right) \tag{2.4}$$

and the grand-canonical particle densities verify

$$\begin{aligned}\rho_\Lambda^X(\beta,\alpha) &\equiv \left\langle \frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^X}(\beta,\alpha) = \partial_\alpha p_\Lambda^X(\beta,\alpha)\\ \rho_\Lambda^{SX}(\beta,\mu) &\equiv \left\langle \frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^{SX}}(\beta,\mu) = \partial_\mu p_\Lambda^{SX}(\beta,\mu).\end{aligned} \tag{2.5}$$

Here $\langle - \rangle_{H_\Lambda^X}(\beta,\alpha)$ and $\langle - \rangle_{H_\Lambda^{SX}}(\beta\mu)$ represent the (finite volume) *grand-canonical* Gibbs state for some Hamiltonian H_Λ^X or H_Λ^{SX} and we also define by $\langle - \rangle_{H_\Lambda^X}(\beta,\rho)$ and $\langle - \rangle_{H_\Lambda^{SX}}(\beta,\rho)$ the corresponding (finite volume) *canonical* Gibbs states (see (2.3)):

$$\begin{aligned}\langle - \rangle_{H_\Lambda^X}(\beta,\rho) &\equiv \frac{\mathrm{Tr}_{\mathcal{H}_B^{(n)}}\left(\left\{(-)\,\mathrm{e}^{-\beta H_\Lambda^X}\right\}^{(n)}\right)}{\mathrm{Tr}_{\mathcal{H}_B^{(n)}}\left(\left\{\mathrm{e}^{-\beta H_\Lambda^X}\right\}^{(n)}\right)} \qquad \langle - \rangle_{H_\Lambda^{SX}}(\beta,\rho) \equiv \frac{\mathrm{Tr}_{\mathcal{H}_B^{(n)}}\left(\left\{(-)\,\mathrm{e}^{-\beta H_\Lambda^{SX}}\right\}^{(n)}\right)}{\mathrm{Tr}_{\mathcal{H}_B^{(n)}}\left(\left\{\mathrm{e}^{-\beta H_\Lambda^{SX}}\right\}^{(n)}\right)}\\ \langle - \rangle_{H_\Lambda^X}(\beta,\alpha) &\equiv \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left((-)\,\mathrm{e}^{-\beta(H_\Lambda^X-\alpha N_\Lambda)}\right)}{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^X-\alpha N_\Lambda)}\right)} \qquad \langle - \rangle_{H_\Lambda^{SX}}(\beta,\mu) \equiv \frac{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left((-)\,\mathrm{e}^{-\beta(H_\Lambda^{SX}-\mu N_\Lambda)}\right)}{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^{SX}-\mu N_\Lambda)}\right)}.\end{aligned} \tag{2.6}$$

Now, the thermodynamic limits of functions (2.2), (2.4) and (2.5) verify:

Conditions 2.1.

(i) The (infinite volume) free-energy density

$$f^X(\beta,\rho) \equiv \lim_\Lambda f_\Lambda^X(\beta,\rho) < +\infty \tag{2.7}$$

is defined for any $\beta > 0$ and $\rho > 0$.

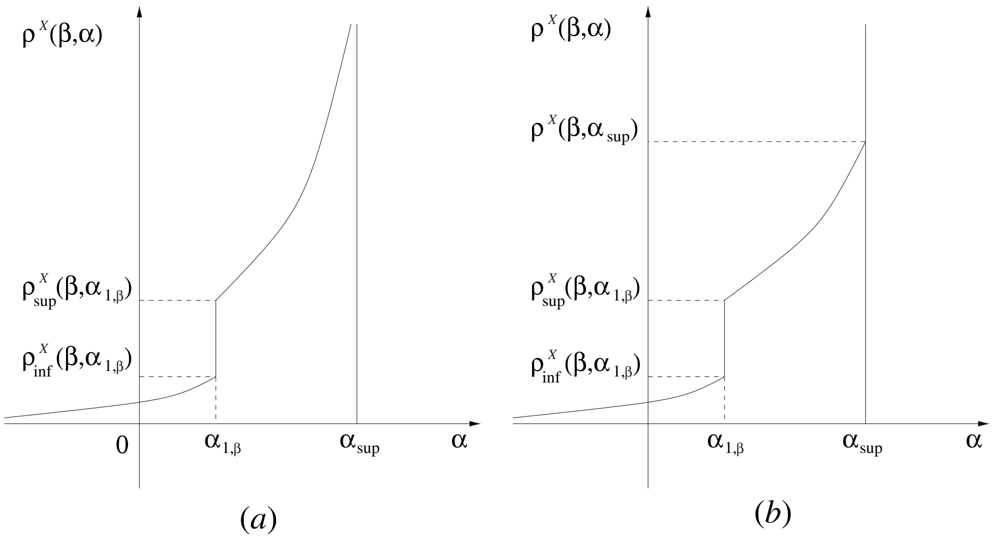

Figure 1. Illustration of the (infinite volume) particle density $\rho^X(\beta, \alpha)$ (2.11) considering the existence of $\alpha_{1,\beta}$ with (*a*) no saturation of the particle density $\rho^X(\beta, \alpha)$, see (2.16); (*b*) saturation of the particle density $\rho^X(\beta, \alpha)$, see (2.17).

(ii) The stability domain Q^X of H^X_Λ is equal to

$$Q^X \equiv \left\{(\beta > 0, \alpha \in \mathbb{R}) : \lim_\Lambda p^X_\Lambda(\beta, \alpha) < +\infty\right\} = Q \equiv \{\beta > 0\} \times \{\alpha < \alpha_{\text{sup}} < +\infty\}. \tag{2.8}$$

(iii) Fixing the inverse temperature $\beta > 0$, the thermodynamic limit of $p^X_\Lambda(\beta, \alpha)$ (2.4), i.e.

$$p^X(\beta, \alpha) \equiv \lim_\Lambda p^X_\Lambda(\beta, \alpha) < +\infty \qquad \text{for} \quad \alpha < \alpha_{\text{sup}} \tag{2.9}$$

and the (infinite volume) free-energy density $f^X(\beta, \rho)$ (2.7) are always related by the Legendre transformation,

$$\begin{aligned} p^X(\beta, \alpha) &= \sup_{\rho>0}\{\alpha\rho - f^X(\beta, \rho)\} \qquad \alpha < \alpha_{\text{sup}} \\ f^X(\beta, \rho) &= \sup_{\alpha<\alpha_{\text{sup}}}\{\alpha\rho - p^X(\beta, \alpha)\} \qquad \rho > 0 \end{aligned} \tag{2.10}$$

i.e. the weak equivalence of canonical and grand-canonical ensembles is verified for the gas X (1.17).

(iv) The (infinite volume) particle density (see figure 1)

$$\rho^X(\beta, \alpha) \equiv \lim_\Lambda \rho^X_\Lambda(\beta, \alpha) \tag{2.11}$$

is a continuous function for $\alpha < \alpha_{\text{sup}}$, except for one chemical potential $\alpha_{1,\beta} < \alpha_{\text{sup}}$.

For further discussions concerning the assumptions of conditions 2.1, see [1]. If

$$\lim_{\alpha\to\alpha_{\text{sup}}^-} p^X(\beta, \alpha) < +\infty \tag{2.12}$$

then we extend $p^X(\beta, \alpha)$ by continuity to

$$Q^X \cup \{(\beta > 0, \alpha_{\text{sup}})\} = Q \cup \{(\beta > 0, \alpha_{\text{sup}})\} = \{\beta > 0\} \times \{\alpha \leqslant \alpha_{\text{sup}} < +\infty\}$$

i.e.

$$p^X(\beta, \alpha_{\sup}) \equiv \lim_{\alpha \to \alpha_{\sup}^-} p^X(\beta, \alpha) < +\infty. \tag{2.13}$$

Note that $\{p_\Lambda^X(\beta, \alpha)\}_\Lambda$ is a set of convex functions for $\alpha < \alpha_{\sup}$ and using the Griffiths lemma [15, 16], (2.5) combined with (2.11) implies in the thermodynamic limit

$$\rho^X(\beta, \alpha) = \partial_\alpha p^X(\beta, \alpha). \tag{2.14}$$

Note also that if there is discontinuity of the particle density $\rho^X(\beta, \alpha)$ (2.11) for some $\alpha = \alpha_{1,\beta}$ (condition 2.1 (iv)) we have

$$\begin{aligned}
\rho_{\inf}^X(\beta, \alpha_{1,\beta}) &\equiv \lim_{\alpha \to \alpha_{1,\beta}^-} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{1,\beta}^-} \partial_\alpha p^X(\beta, \alpha) < \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha) \\
\rho_{\sup}^X(\beta, \alpha_{1,\beta}) &\equiv \lim_{\alpha \to \alpha_{1,\beta}^+} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{1,\beta}^+} \partial_\alpha p^X(\beta, \alpha) < \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha)
\end{aligned} \tag{2.15}$$

cf (2.14). Moreover, we may have two different cases for $\alpha \to \alpha_{\sup}^-$: either one has

$$\lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{\sup}^-} \partial_\alpha p^X(\beta, \alpha) = +\infty \tag{2.16}$$

or there is a *saturation* of the infinite volume particle density $\rho^X(\beta, \alpha)$ (2.11), i.e. one has a critical particle density

$$\rho^X(\beta, \alpha_{\sup}) \equiv \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha) = \lim_{\alpha \to \alpha_{\sup}^-} \partial_\alpha p^X(\beta, \alpha) < +\infty \tag{2.17}$$

cf (2.14).

2.2. *Summary of basic thermodynamic results*

Now, we quickly sum up the thermodynamic relations between the Bose systems X (1.17) and SX (1.18) proved in [1].

(1) In the canonical ensemble, by (2.1) we have

$$f^{SX}(\beta, \rho) \equiv \lim_\Lambda f_\Lambda^{SX}(\beta, \rho) = f^X(\beta, \rho) + \lambda\rho^2 \qquad \rho > 0 \tag{2.18}$$

cf (2.2) and (2.7).

(2) In the grand-canonical ensemble, the thermodynamic limit $p^{SX}(\beta, \mu)$ of $p_\Lambda^{SX}(\beta\mu)$ (2.4) is deduced from $p^X(\beta, \alpha)$ (2.9) by

$$p^{SX}(\beta, \mu) = \inf_{\alpha < \alpha_{\sup}} \left\{ p^X(\beta, \alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} = p^X(\beta, \widetilde{\alpha}_\beta(\mu)) + \frac{(\mu - \widetilde{\alpha}_\beta(\mu))^2}{4\lambda} \tag{2.19}$$

for $(\beta, \mu) \in Q^S$ (1.7). Note that $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{\sup}$ (2.19) is an increasing continuous function for $\mu \in \mathbb{R}$ (see figures 2 and 3). Moreover, the infinite volume particle density $\rho^{SX}(\beta, \mu)$ is a continuous and strictly increasing function from $\mu \in \mathbb{R}$ to $(0, +\infty)$ and verifies

$$\rho^{SX}(\beta, \mu) \equiv \lim_\Lambda \rho_\Lambda^{SX}(\beta, \mu) = \partial_\mu p^{SX}(\beta, \mu) = \frac{(\mu - \widetilde{\alpha}_\beta(\mu))}{2\lambda} \tag{2.20}$$

for $(\beta, \mu) \in Q^S$ (1.7) (see figures 4 and 5). Actually, one has

$$\rho^{SX}(\beta, \mu) = \rho^X(\beta, \widetilde{\alpha}_\beta(\mu)) \tag{2.21}$$

for

$$\mu \in I_\mu(\beta) \equiv \{\mu \in \mathbb{R} : \widetilde{\alpha}_\beta(\mu) \in (-\infty, \alpha_{\sup}), \widetilde{\alpha}_\beta(\mu) \neq \alpha_{1,\beta}\} \tag{2.22}$$

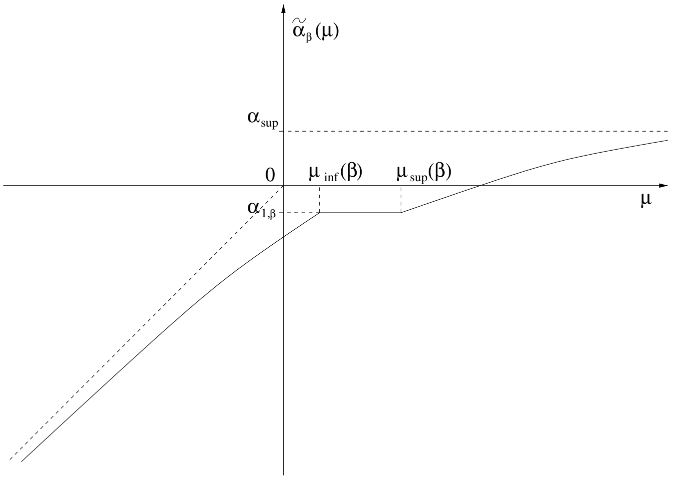

Figure 2. Illustration of the function $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{sup}$ defined by equation (2.19) with no saturation of the particle density $\rho^X(\beta, \alpha)$ (2.11), see (2.16).

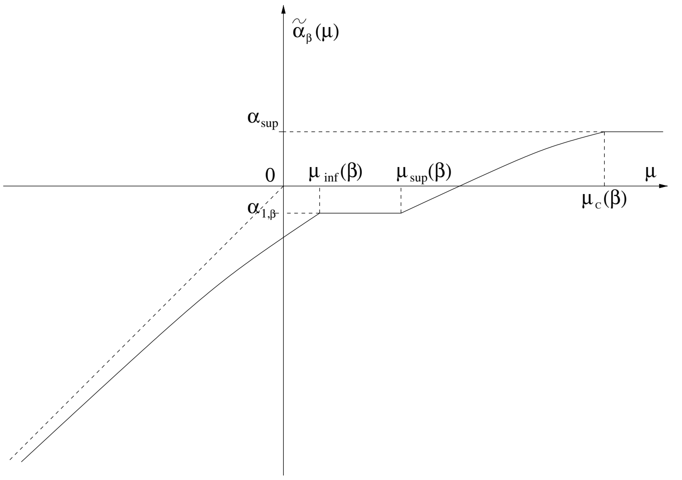

Figure 3. Illustration of the function $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{sup}$ defined by equation (2.19) with saturation of the particle density $\rho^X(\beta, \alpha)$ (2.11), see (2.17).

i.e. for

$$\mu < \lim_{\alpha \to \alpha_{sup}^-} 2\lambda \rho^X(\beta, \alpha) + \alpha_{sup} \qquad \text{with} \quad \mu \notin (\mu_{1,inf}(\beta), \mu_{1,sup}(\beta)). \tag{2.23}$$

Here, if $\alpha_{1,\beta}$ exists (condition 2.1 (iv)), the two chemical potentials $\mu_{1,inf}(\beta)$ and $\mu_{1,sup}(\beta)$ are defined by

$$\begin{aligned} \mu_{1,inf}(\beta) &\equiv 2\lambda \rho_{inf}^X(\beta, \alpha_{1,\beta}) + \alpha_{1,\beta} = 2\lambda \lim_{\alpha \to \alpha_{1,\beta}^-} \partial_\alpha p^X(\beta, \alpha) + \alpha_{1,\beta} \\ \mu_{1,sup}(\beta) &\equiv 2\lambda \rho_{sup}^X(\beta, \alpha_{1,\beta}) + \alpha_{1,\beta} = 2\lambda \lim_{\alpha \to \alpha_{1,\beta}^+} \partial_\alpha p^X(\beta, \alpha) + \alpha_{1,\beta} \end{aligned} \tag{2.24}$$

cf (2.15), whereas, if there is a critical particle density $\rho^X(\beta, \alpha_{sup})$ (2.17) then we define by

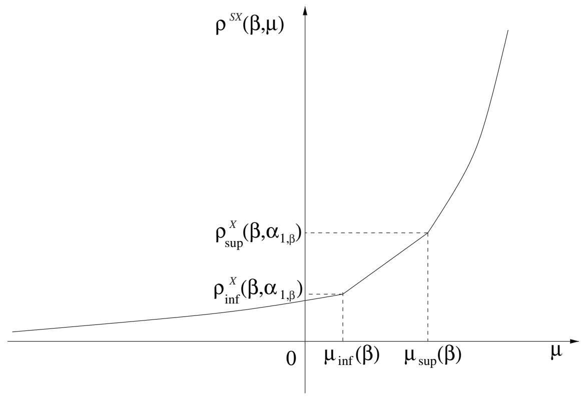

Figure 4. Illustration of the particle density $\rho^{SX}(\beta,\mu)$ (2.20) with no saturation of the particle density $\rho^X(\beta,\alpha)$.

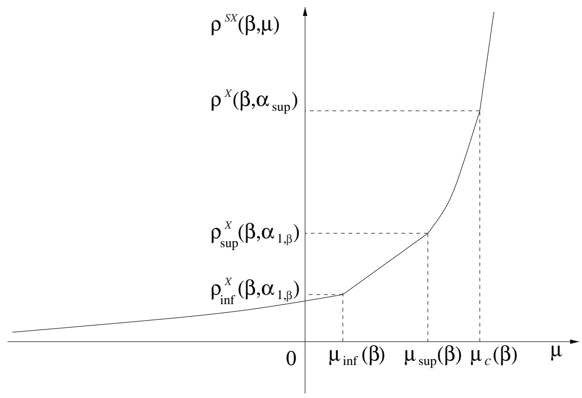

Figure 5. Illustration of the particle density $\rho^{SX}(\beta,\mu)$ (2.20) with saturation of the particle density $\rho^X(\beta,\alpha)$.

$$\mu_c(\beta) \equiv 2\lambda\rho^X(\beta,\alpha_{\sup}) + \alpha_{\sup} = 2\lambda \lim_{\alpha\to\alpha_{\sup}^-} \partial_\alpha p^X(\beta,\alpha) + \alpha_{\sup} \tag{2.25}$$

the corresponding critical chemical potential.

Remark 2.2. For $\mu \in I_\mu(\beta)$ (2.22) we have

$$\partial_\alpha \left\{ p^X(\beta,\alpha) + \frac{(\mu-\alpha)^2}{4\lambda} \right\}\Big|_{\alpha=\widetilde{\alpha}_\beta(\mu)} = \left\{ \rho^X(\beta,\alpha) - \frac{\mu-\alpha}{2\lambda} \right\}\Big|_{\alpha=\widetilde{\alpha}_\beta(\mu)} = 0$$

with $\widetilde{\alpha}_\beta(\mu) \leqslant \alpha_{\sup}$ defined by (2.19).

(3) Now, let us consider by $\rho > 0$ the *fixed* particle density in the *grand-canonical* ensemble. If the particle density $\rho^X(\beta,\alpha)$ (2.11) is a strictly increasing function for $\alpha < \alpha_{\sup}$, then for any

$$\rho < \lim_{\alpha\to\alpha_{\sup}^-} \rho^X(\beta,\alpha) \tag{2.26}$$

there is a unique $\alpha(\rho) < \alpha_{\sup}$ such that

$$\rho^X(\beta,\alpha(\rho)) = \partial_\alpha p^X(\beta,\alpha(\rho)) = \rho \tag{2.27}$$

cf (2.14). If there is a saturation (2.17) of the infinite volume particle density $\rho^X(\beta,\alpha)$ (2.11), we extend the function $\alpha(\rho)$ to $\rho \geqslant \rho^X(\beta,\alpha_{\sup})$ by

$$\alpha(\rho) \equiv \alpha_{\sup} \qquad \text{for} \quad \rho \geqslant \rho^X(\beta,\alpha_{\sup}). \tag{2.28}$$

For the model SX (1.18), the unique chemical potential $\mu(\rho)$ is a solution of the equation

$$\rho^{SX}(\beta,\mu(\rho)) = \rho \tag{2.29}$$

and also verifies

$$\widetilde{\alpha}_\beta(\mu(\rho)) = \alpha(\rho) \tag{2.30}$$

for any $\rho > 0$. In fact, through (2.20) combined with (2.30) we have

$$\mu(\rho) = 2\lambda\rho + \alpha(\rho) \qquad \rho > 0. \tag{2.31}$$

If we consider the grand-canonical pressures (2.4) in the thermodynamic limit for a fixed particle density $\rho > 0$, then one gets

$$p^{SX}(\beta,\mu(\rho)) = p^X(\beta,\widetilde{\alpha}_\beta(\mu(\rho))) + \lambda\rho^2 = p^X(\beta,\alpha(\rho)) + \lambda\rho^2. \tag{2.32}$$

Note that (2.32) is related to (2.18), see discussions in [1].

Remark 2.3. For any $\rho > 0$ the function $\rho \to \mu(\rho)$ (2.29) is bijective from $\rho > 0$ to $\mu(\rho) \in \mathbb{R}$ whereas, if (2.17) is satisfied, the function $\rho \to \alpha(\rho)$ (2.27)–(2.28) may be bijective only from $\rho \leqslant \rho^X(\beta,\alpha_{\sup})$ to $\alpha(\rho) \leqslant \alpha_{\sup}$.

3. Bose condensations in the grand-canonical ensemble

In the last section (see also [1]), using the thermodynamic behaviour of the first non-superstable model X (1.17), we recall the thermodynamic limits of basic thermodynamic functions (2.2)–(2.5) for the superstable model SX (1.18), i.e. the free-energy density (2.18), the grand-canonical pressure (2.19) (see also (2.32)) and particle density (2.20)–(2.25). The purpose of this section is now to analyse the relations between the existence of a Bose condensation in the model X and the appearance of a similar Bose condensation for the corresponding superstable model SX. In this section, note that we restrict our arguments only in the *grand-canonical* ensemble: (β,α) for the (non-superstable) model X (1.17) and (β,μ) for the (superstable) model SX (1.18).

3.1. Existence of the same 'global' Bose condensation for a fixed particle density

First, we recall that formally we may have six kinds of condensation: non-conventional or conventional, of type I, II or III, see appendix A. More generally, the existence of one or several kinds of Bose condensation for the non-superstable model X (1.17) means that at least we have

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}}\langle N_k\rangle_{H_\Lambda^X}(\beta,\alpha_\Lambda(\rho)) > 0 \tag{3.1}$$

i.e. a 'global' Bose condensation for a *fixed* particle density $\rho > 0$ in the grand-canonical ensemble (β,α). Here $N_k \equiv a_k^* a_k$ and $\alpha_\Lambda(\rho)$ is defined by

$$\rho_\Lambda^X(\beta,\alpha_\Lambda(\rho)) = \partial_\alpha p_\Lambda^X(\beta,\alpha_\Lambda(\rho)) = \rho \tag{3.2}$$

for any $\rho > 0$. In order to simplify our arguments, here we *assume* that

$$\sup_{\alpha\in Q_{\Lambda,\beta}^X}\rho_\Lambda^X(\beta,\alpha) = +\infty \qquad Q_{\Lambda,\beta}^X \equiv \left\{\alpha\in\mathbb{R} : \rho_\Lambda^X(\beta,\alpha) < +\infty\right\}. \tag{3.3}$$

In fact, if

$$\sup_{\alpha\in Q^X_{\Lambda,\beta}} \rho^X_\Lambda(\beta,\alpha) \equiv \rho_{c,\Lambda}(\beta) < +\infty \tag{3.4}$$

then we should take throughout this subsection a fixed particle density

$$\rho \leqslant \lim_\Lambda \rho_{c,\Lambda}(\beta).$$

Remark 3.1. In spite of condition 2.1 which implies

$$\lim_\Lambda \rho^X_\Lambda(\beta,\alpha > \alpha_{\sup}) = +\infty$$

note that for a specific set of $\alpha > \alpha_{\sup}$ we may have

$$\rho^X_\Lambda(\beta,\alpha > \alpha_{\sup}) < +\infty. \tag{3.5}$$

An example is given below by the diagonal model (3.40) studied in paper [12] which verifies (3.5) for $\alpha \in \mathbb{R}$ even if α has to be smaller than $\alpha_{\sup} = 0$ in order to get a finite pressure or particle density in the thermodynamic limit.

Note that for any $\rho > 0$,

$$\alpha(\rho) = \lim_\Lambda \alpha_\Lambda(\rho) \tag{3.6}$$

with $\alpha(\rho)$ defined by (2.27)–(2.28). We also denote by $\mu_\Lambda(\rho)$ the chemical potential satisfying

$$\rho^{SX}_\Lambda(\beta,\mu_\Lambda(\rho)) = \rho \tag{3.7}$$

and via (2.29) one has

$$\mu(\rho) = \lim_\Lambda \mu_\Lambda(\rho). \tag{3.8}$$

In order to control the 'global' Bose condensation in the *superstable* model SX (1.18) we introduce the auxiliary Hamiltonians

$$H^X_{\Lambda,\gamma} \equiv H^X_\Lambda - \gamma \sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}} a^*_k a_k \tag{3.9}$$

$$H^{SX}_{\Lambda,\gamma} \equiv H^{SX}_\Lambda - \gamma \sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}} a^*_k a_k = H^X_{\Lambda,\gamma} + \frac{\lambda}{V} N^2_\Lambda \tag{3.10}$$

for fixed $\delta > 0$, $\lambda > 0$ and $\gamma < \varepsilon_\delta \equiv \varepsilon_{\|k\|=\delta}$. Note that relation (2.1) verified by H^X_Λ (1.17) is also satisfied by the stable Hamiltonian $H^X_{\Lambda,\gamma}$ (3.9). Then we define their corresponding grand-canonical pressures

$$p^X_\Lambda(\beta,\alpha,\gamma) \equiv \frac{1}{\beta V} \ln \operatorname{Tr}_{\mathcal{F}^B_\Lambda}\left(\mathrm{e}^{-\beta(H^X_{\Lambda,\gamma}-\alpha N_\Lambda)}\right) \tag{3.11}$$

$$p^{SX}_\Lambda(\beta,\mu,\gamma) \equiv \frac{1}{\beta V} \ln \operatorname{Tr}_{\mathcal{F}^B_\Lambda}\left(\mathrm{e}^{-\beta(H^{SX}_{\Lambda,\gamma}-\mu N_\Lambda)}\right). \tag{3.12}$$

In order to use (2.19), the Hamiltonian $H^X_{\Lambda,\gamma}$ (3.9) should verify some sufficient conditions as (i)–(iii) of conditions 2.1 (see also discussions in [1]). The Hamiltonian $H^X_{\Lambda,\gamma}$ (3.9) could be seen as H^X_Λ (1.17) modulo the new free-particle spectrum transformation

$$\varepsilon_k \to \varepsilon_{k,\gamma} \equiv \varepsilon_k - \gamma \cdot \chi_{[\delta,+\infty)} \qquad (\|k\|) \geqslant 0 \qquad \gamma < \varepsilon_\delta \equiv \varepsilon_{\|k\|=\delta} \qquad \delta > 0 \tag{3.13}$$

where $\chi_A(x)$ is the characteristic function of a domain A. In fact the assumptions (i) and (iii) of conditions 2.1 are stable modulo a free-particle spectrum transformation (3.13). In particular, to prove (2.10) for (β,α) in the stability domain Q^X_γ of $H^X_{\Lambda,\gamma}$ (3.9), i.e.

$$(\beta, \alpha) \in Q^X_\gamma \equiv \left\{(\beta > 0, \alpha \in \mathbb{R}) : \lim_\Lambda p^X_\Lambda(\beta, \alpha, \gamma) < +\infty\right\} \neq \{\varnothing\} \tag{3.14}$$

the difficulty comes from the interaction term U^X_Λ (1.17) and not from the kinetic part T_Λ (1.2) or here from

$$T_{\Lambda,\gamma} = \sum_{k\in\Lambda^*} \varepsilon_{k,\gamma} a^*_k a_k \tag{3.15}$$

as soon as $\varepsilon_{k,\gamma} > 0$ for $k \in \Lambda^*\backslash\{0\}$, see for example [11]. Then the Hamiltonian $H^X_{\Lambda,\gamma}$ (3.9) also verifies (i) and (iii) of conditions 2.1. The stability domain Q^X_γ (3.14) is

$$Q^X_\gamma = \{\beta > 0\} \times \{\alpha < \alpha_{\sup}(\gamma) < +\infty\} \tag{3.16}$$

see (2.8). Note that $\alpha_{\sup}(\gamma) \in [\alpha_{\sup} - \gamma, \alpha_{\sup}]$ if $\gamma > 0$.

For a large class of known Hamiltonians, as soon as $\varepsilon_{k,\gamma} > 0$ for $k \in \Lambda^*\backslash\{0\}$, note that the stability domain Q^X (2.8) is in general stable modulo the free-particle spectrum transformation (3.13), i.e.

$$Q^X = Q^X_\gamma = Q \equiv \{\beta > 0\} \times \{\alpha < \alpha_{\sup}(\gamma) = \alpha_{\sup} < +\infty\} \tag{3.17}$$

for $\gamma < \varepsilon_\delta$ and $\delta > 0$ sufficiently small, see (2.8), (3.14) and (3.16).

Thus the Hamiltonian $H^X_{\Lambda,\gamma}$ (3.9) verifies (i) and (iii) of conditions 2.1, see also (3.16), and equality (2.19) remains unchanged,

$$\begin{aligned} p^{SX}(\beta, \mu, \gamma) \equiv \lim_\Lambda p^{SX}_\Lambda(\beta, \mu, \gamma) &= \inf_{\alpha<\alpha_{\sup}(\gamma)} \left\{ p^X(\beta, \alpha, \gamma) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \\ &= p^X(\beta, \widetilde{\alpha}_{\beta,\gamma}(\mu), \gamma) + \frac{(\mu - \widetilde{\alpha}_{\beta,\gamma}(\mu))^2}{4\lambda} \end{aligned} \tag{3.18}$$

for $(\beta, \mu) \in Q^S$ (1.7), where $p^X(\beta, \alpha, \gamma)$ is the pressure $p^X(\beta, \alpha)$ (2.9) with the free-particle spectrum (3.13).

Theorem 3.2. *Let us consider a model X (1.17) satisfying conditions 2.1. Then, with $\alpha_\Lambda(\rho)$ and $\mu_\Lambda(\rho)$ defined by (3.2) and (3.7) respectively, one has*

$$\lim_{\delta\to 0^+} \lim_\Lambda \left\{ \frac{1}{V} \sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}} \langle N_k\rangle_{H^{SX}_\Lambda}(\beta, \mu_\Lambda(\rho)) - \frac{1}{V} \sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}} \langle N_k\rangle_{H^X_\Lambda}(\beta, \alpha_\Lambda(\rho)) \right\} = 0 \tag{3.19}$$

for $\rho \leqslant \lim_{\alpha\to\alpha^-_{\sup}} \rho^X(\beta, \alpha)$ and $\rho \notin \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right)$ (2.15) if we consider the existence of $\alpha_{1,\beta}$ (condition 2.1 (iv)).

Considering that, for $\delta > 0$ sufficiently small, the particle density associated with $H^X_{\Lambda,\gamma}$ (3.9),

$$\rho^X(\beta, \alpha, \gamma) \equiv \lim_\Lambda \left\langle \frac{N_\Lambda}{V} \right\rangle_{H^X_{\Lambda,\gamma}} (\beta, \alpha) = \partial_\alpha \left\{ \lim_\Lambda p^X_\Lambda(\beta, \alpha, \gamma) \right\} = \partial_\alpha p^X(\beta, \alpha, \gamma) \tag{3.20}$$

(see (3.11)) is a continuous function of $\gamma < \varepsilon_\delta$, the double limit (3.19) is also verified:

- *for $\rho > \rho^X(\beta, \alpha_{\sup})$ if there is a critical particle density $\rho^X(\beta, \alpha_{\sup})$ (2.17) and one has (3.17) $\alpha_{\sup}(\gamma) = \alpha_{\sup}$;*
- *for $\rho \in \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right)$ (2.15), if the point $\alpha_{1,\beta}(\gamma) < \alpha_{\sup}(\gamma)$ of discontinuity of $\rho^X(\beta, \alpha, \gamma)$ (3.20) is $\alpha_{1,\beta}(\gamma) = \alpha_{1,\beta}$.*

Proof.

(1) Let $\delta > 0$, then we have

$$\frac{1}{V}\sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}}\langle N_k\rangle_{H^X_\Lambda}(\beta,\alpha) = \rho^X_\Lambda(\beta,\alpha) - \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}\langle N_k\rangle_{H^X_\Lambda}(\beta,\alpha)$$
$$\frac{1}{V}\sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}}\langle N_k\rangle_{H^{SX}_\Lambda}(\beta,\mu) = \rho^{SX}_\Lambda(\beta,\mu) - \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}\langle N_k\rangle_{H^{SX}_\Lambda}(\beta,\mu). \qquad (3.21)$$

(2) Note that via (3.18)

$$\partial_\gamma p^{SX}(\beta,\mu(\rho),\gamma) = \partial_\alpha\left\{p^X(\beta,\alpha,\gamma) + \frac{(\mu(\rho)-\alpha)^2}{4\lambda}\right\}\bigg|_{\alpha=\widetilde{\alpha}_{\beta,\gamma}(\mu(\rho))}\partial_\gamma\widetilde{\alpha}_{\beta,\gamma}(\mu(\rho))$$
$$+\,\partial_\gamma\left\{p^X(\beta,\alpha,\gamma) + \frac{(\mu(\rho)-\alpha)^2}{4\lambda}\right\}\bigg|_{\alpha=\widetilde{\alpha}_{\beta,\gamma}(\mu(\rho))} \qquad (3.22)$$

for $\delta > 0$ sufficiently small. Via (2.30), (3.2) and (3.6)–(3.8) note that

$$\alpha(\rho) = \lim_\Lambda \alpha_\Lambda(\rho) = \widetilde{\alpha}_\beta(\mu(\rho)) = \widetilde{\alpha}_{\beta,\gamma=0}(\mu(\rho))$$
$$\mu(\rho) = \lim_\Lambda \mu_\Lambda(\rho). \qquad (3.23)$$

(3) Let

$$\rho \leqslant \lim_{\alpha\to\alpha_{\sup}^-}\rho^X(\beta,\alpha)$$

i.e.

$$\mu(\rho) \leqslant 2\lambda \lim_{\alpha\to\alpha_{\sup}^-}\rho^X(\beta,\alpha) + \alpha_{\sup}$$

see (2.31) and considering the existence of $\alpha_{1,\beta}$ (condition 2.1 (iv)) we take $\rho \notin \left(\rho^X_{\inf}(\beta,\alpha_{1,\beta}), \rho^X_{\sup}(\beta,\alpha_{1,\beta})\right)$ (2.15), i.e. $\mu(\rho) \notin (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$ (2.24), cf (2.31). Since for $\rho \leqslant \lim_{\alpha\to\alpha_{\sup}^-}\rho^X(\beta,\alpha)$ and $\rho \notin \left(\rho^X_{\inf}(\beta,\alpha_{1,\beta}), \rho^X_{\sup}(\beta,\alpha_{1,\beta})\right)$, one has

$$\partial_\alpha\left\{p^X(\beta,\alpha) + \frac{(\mu(\rho)-\alpha)^2}{4\lambda}\right\}\bigg|_{\alpha=\widetilde{\alpha}_\beta(\mu(\rho))}$$
$$= \partial_\alpha\left\{p^X(\beta,\alpha,\gamma) + \frac{(\mu(\rho)-\alpha)^2}{4\lambda}\right\}\bigg|_{\alpha=\widetilde{\alpha}_\beta(\mu(\rho)),\gamma=0} = 0$$

see remark 2.2, by (3.23) combined with (3.22) we obtain

$$\partial_\gamma p^{SX}(\beta,\mu(\rho),\gamma)|_{\gamma=0} = \partial_\gamma p^X(\beta,\widetilde{\alpha}_\beta(\mu(\rho)),\gamma)|_{\gamma=0} \qquad (3.24)$$

for $\rho \leqslant \lim_{\alpha\to\alpha_{\sup}^-}\rho^X(\beta,\alpha)$ and $\rho \notin \left(\rho^X_{\inf}(\beta,\alpha_{1,\beta}), \rho^X_{\sup}(\beta,\alpha_{1,\beta})\right)$ (2.15).

(4) Considering the existence of a critical particle density $\rho^X(\beta,\alpha_{\sup})$ (2.17), let $\rho > \rho^X(\beta,\alpha_{\sup})$, i.e.

$$\mu(\rho) > \mu(\rho^X(\beta,\alpha_{\sup})) = 2\lambda\rho^X(\beta,\alpha_{\sup}) + \alpha_{\sup} = \mu_c(\beta) \qquad (3.25)$$

cf (2.25) and (2.31). Via (2.17) combined with the continuity of $\rho^X(\beta,\alpha,\gamma)$ (3.20) as a function of $\gamma < \varepsilon_\delta$, there is also a critical density

$$\rho^X(\beta,\alpha_{\sup}(\gamma),\gamma) \equiv \lim_{\alpha\to\alpha_{\sup}^-(\gamma)}\rho^X(\beta,\alpha,\gamma) < +\infty.$$

Following the same arguments used to study the function $\widetilde{\alpha}_\beta(\mu)$ (2.19) (cf [1]), we get that $\widetilde{\alpha}_{\beta,\gamma}(\mu) = \alpha_{\sup}(\gamma)$ (3.18) is constant as a function of

$$\mu > \mu_c(\beta,\gamma) \equiv 2\lambda\rho^X(\beta,\alpha_{\sup}(\gamma),\gamma) + \alpha_{\sup}(\gamma). \qquad (3.26)$$

Thus, assuming (3.17), we find

$$\partial_\gamma \widetilde{\alpha}_{\beta,\gamma}(\mu) = \partial_\gamma \alpha_{\sup}(\gamma) = 0 \qquad \text{for} \quad \mu > \mu_c(\beta,\gamma) \text{ (3.26)}. \tag{3.27}$$

If $\rho > \rho^X(\beta, \alpha_{\sup})$ then for $|\gamma|$ sufficiently small, $\rho > \rho^X(\beta, \alpha_{\sup}(\gamma), \gamma)$ and by (2.31) $\mu(\rho) > \mu_c(\beta,\gamma)$ (3.26). So, by (3.27)

$$\partial_\gamma \widetilde{\alpha}_{\beta,\gamma}(\mu(\rho)) = 0 \qquad \text{for} \quad \rho > \rho^X(\beta, \alpha_{\sup}) \text{ (2.17)} \tag{3.28}$$

and via (3.23) equation (3.22) implies

$$\partial_\gamma p^{SX}(\beta, \mu(\rho), \gamma)|_{\gamma=0} = \partial_\gamma p^X(\beta, \widetilde{\alpha}_\beta(\mu(\rho)), \gamma)|_{\gamma=0} \tag{3.29}$$

for $\rho > \rho^X(\beta, \alpha_{\sup})$ (2.17).

(5) Let $\rho \in \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right)$ (2.15), i.e. $\mu(\rho) \in (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$ (2.24), cf (2.31). Then, there is $\alpha_{1,\beta}(\gamma) < \alpha_{\sup}(\gamma)$ such that $\rho^X(\beta,\alpha,\gamma)$, as a function of $\alpha < \alpha_{\sup}(\gamma)$, is discontinuous on $\alpha_{1,\beta}(\gamma)$,

$$\begin{aligned} \rho^X_{\inf}(\beta, \alpha_{1,\beta}(\gamma), \gamma) &\equiv \lim_{\alpha \to \alpha^-_{1,\beta}(\gamma)} \rho^X(\beta,\alpha,\gamma) < \lim_{\alpha \to \alpha^-_{\sup}(\gamma)} \rho^X(\beta,\alpha,\gamma) \\ \rho^X_{\sup}(\beta, \alpha_{1,\beta}(\gamma), \gamma) &\equiv \lim_{\alpha \to \alpha^+_{1,\beta}(\gamma)} \rho^X(\beta,\alpha,\gamma) < \lim_{\alpha \to \alpha^-_{\sup}(\gamma)} \rho^X(\beta,\alpha,\gamma) \end{aligned} \tag{3.30}$$

with

$$\begin{aligned} &\alpha_{1,\beta}(0) = \alpha_{1,\beta} \\ &\rho^X_{\inf}(\beta, \alpha_{1,\beta}(0), 0) = \rho^X_{\inf}(\beta, \alpha_{1,\beta}) \\ &\rho^X_{\sup}(\beta, \alpha_{1,\beta}(0), 0) = \rho^X_{\sup}(\beta, \alpha_{1,\beta}) \end{aligned}$$

and $\rho^X_{\inf}(\beta, \alpha_{1,\beta}(\gamma), \gamma) < \rho^X_{\sup}(\beta, \alpha_{1,\beta}(\gamma), \gamma)$. Again, following the same arguments used to study the function $\widetilde{\alpha}_\beta(\mu)$ (2.19) (cf [1]), $\widetilde{\alpha}_{\beta,\gamma}(\mu) = \alpha_{1,\beta}(\gamma)$ (3.18) is *constant* as a function of $\mu \in [\mu_{1,\inf}(\beta,\gamma), \mu_{1,\sup}(\beta,\gamma)]$. Here

$$\begin{aligned} \mu_{1,\inf}(\beta,\gamma) &\equiv 2\lambda \rho^X_{\inf}(\beta, \alpha_{1,\beta}(\gamma), \gamma) + \alpha_{1,\beta}(\gamma) \\ \mu_{1,\sup}(\beta,\gamma) &\equiv 2\lambda \rho^X_{\sup}(\beta, \alpha_{1,\beta}(\gamma), \gamma) + \alpha_{1,\beta}(\gamma) \end{aligned} \tag{3.31}$$

see (3.30). Then, assuming that $\alpha_{1,\beta}(\gamma) = \alpha_{1,\beta}$ for $\gamma < \varepsilon_\delta$ and $\delta > 0$ sufficiently small, we find

$$\partial_\gamma \widetilde{\alpha}_{\beta,\gamma}(\mu) = \partial_\gamma \alpha_{1,\beta}(\gamma) = 0 \qquad \text{for} \quad \mu \in (\mu_{1,\inf}(\beta,\gamma), \mu_{1,\sup}(\beta,\gamma)) \text{ (3.31)}. \tag{3.32}$$

If $\rho \in \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right)$ (2.15), then for $|\gamma|$ sufficiently small,

$$\rho \in \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}(\gamma), \gamma), \rho^X_{\sup}(\beta, \alpha_{1,\beta}(\gamma), \gamma)\right)$$

(3.30) and by (2.31) $\mu(\rho) \in (\mu_{1,\inf}(\beta,\gamma), \mu_{1,\sup}(\beta,\gamma))$ (3.31). Therefore, since by (3.32)

$$\partial_\gamma \widetilde{\alpha}_{\beta,\gamma}(\mu(\rho)) = 0 \qquad \text{for} \quad \rho \in \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right) \text{ (2.15)} \tag{3.33}$$

by (3.23) equation (3.22) implies again

$$\partial_\gamma p^{SX}(\beta, \mu(\rho), \gamma)|_{\gamma=0} = \partial_\gamma p^X(\beta, \widetilde{\alpha}_\beta(\mu(\rho)), \gamma)|_{\gamma=0} \tag{3.34}$$

for $\rho \in \left(\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right)$ (2.15).

(6) Now, since $\left\{p_\Lambda^X(\beta,\alpha,\gamma)\right\}_\Lambda$ and $\left\{p_\Lambda^{SX}(\beta,\mu,\gamma)\right\}_\Lambda$ are two sets of convex functions with

$$\begin{aligned}\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}\langle N_k\rangle_{H^X_{\Lambda,\gamma}}(\beta,\alpha)=\partial_\gamma p_\Lambda^X(\beta,\alpha,\gamma)\\ \frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}\langle N_k\rangle_{H^{SX}_{\Lambda,\gamma}}(\beta,\mu)=\partial_\gamma p_\Lambda^{SX}(\beta,\mu,\gamma)\end{aligned}\tag{3.35}$$

by the Griffiths lemma [15, 16] and via (3.18), (3.23), (3.24), (3.29) and (3.34), one obtains for $\gamma=0$ and $\rho>0$

$$\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}\langle N_k\rangle_{H^{SX}_\Lambda}(\beta,\mu_\Lambda(\rho))=\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*:\|k\|\geqslant\delta\}}\langle N_k\rangle_{H^X_\Lambda}(\beta,\alpha_\Lambda(\rho)).\tag{3.36}$$

Therefore, equalities (3.21) and (3.36) imply (3.19) by taking the limit $\delta\to0^+$. □

Remark 3.3. If there is a critical particle density $\rho^X(\beta,\alpha_{\sup})$ (2.17), to prove (3.19) for $\rho>\rho^X(\beta,\alpha_{\sup})$, we only need $\partial_\gamma\alpha_{\sup}(\gamma)=0$, see (3.27). Whereas, if there is $\alpha_{1,\beta}$ (condition 2.1 (iv)), for $\rho\in\left(\rho^X_{\inf}(\beta,\alpha_{1,\beta}),\rho^X_{\sup}(\beta,\alpha_{1,\beta})\right)$ (2.15) $\partial_\gamma\alpha_{1,\beta}(\gamma)=0$ is the only condition necessary to get (3.19), see (3.32).

Remark 3.4. The condition $\alpha_{\sup}(\gamma)=\alpha_{\sup}$ or more generally $\partial_\gamma\alpha_{\sup}(\gamma)=0$ is verified by a large class of known Hamiltonians. However, the assumption $\alpha_{1,\beta}(\gamma)=\alpha_{1,\beta}$, or more generally $\partial_\gamma\alpha_{1,\beta}(\gamma)=0$, should be more subtle. For example, one may analyse this last assumption using the Bogoliubov weakly imperfect Bose gas (cf equation (3.81) in [18]) for H_Λ^X (1.17), see [10, 19, 20].

Corollary 3.5. *We consider as verified all the assumptions of theorem 3.2, and also*

$$\lim_{\delta\to0^+}\lim_\Lambda\left\{\frac{1}{V}\sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}}\langle N_k\rangle_{H^X_\Lambda}(\beta,\alpha_\Lambda(\rho))\right\}=\rho-\widetilde{\rho}_c(\beta,\rho)>0\tag{3.37}$$

with

$$\widetilde{\rho}_c(\beta,\rho)\leqslant\lim_{\alpha\to\alpha_{\sup}^-}\rho^X(\beta,\alpha).$$

Then via theorem 3.2 one deduces

$$\lim_{\delta\to0^+}\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}}\langle N_k\rangle_{H^{SX}_\Lambda}(\beta,\mu_\Lambda(\rho))=\rho-\widetilde{\rho}_c(\beta,\rho)>0\tag{3.38}$$

with $\alpha_\Lambda(\rho)$ and $\mu_\Lambda(\rho)$ defined by (3.2) and (3.7) respectively.

Therefore by corollary 3.5 a 'global' Bose condensation (3.37) in the model X (1.17) implies a 'global' Bose condensation (3.38) in the model SX (1.18) with *exactly* the same density for a fixed (full) particle density ρ.

Considering now the PBG and the IBG, i.e. $H_\Lambda^X=T_\Lambda$ (1.2) and $H_\Lambda^{SX}=H_\Lambda^{IBG}$ (1.1), the corollary 3.5 implies again (1.12)–(1.16), i.e. in the grand-canonical ensemble

$$\rho_0^{IBG}(\beta,\rho)=\rho_0^{PBG}(\beta,\rho)=\sup\{0,\rho-\rho^{PBG}(\beta,0)\}.\tag{3.39}$$

3.2. Existence of different kinds of Bose condensation

In spite of the existence of the same 'global' Bose condensation for the models X (1.17) and SX (1.18) (corollary 3.5), we recall that the Bose condensation phenomena are more complex than, for example, in the PBG or in the IBG. Indeed formally the 'global' Bose condensation (3.1) or (3.37) may be constituted of *six* kinds of Bose condensation (cf appendix A). For example, let us consider the non-superstable model with *diagonal* interactions presented in [12],

$$H_\Lambda^{BZ} = \varepsilon_0 a_0^* a_0 + \frac{g_0}{V} a_0^* a_0^* a_0 a_0 + \sum_{k\in\Lambda^*\backslash\{0\}} \left\{\varepsilon_k a_k^* a_k + \frac{g}{V} a_k^* a_k^* a_k a_k\right\} \tag{3.40}$$

where

$$\varepsilon_0 < 0 \qquad g_0 > 0$$
$$\varepsilon_{k\neq 0} = \hbar^2 k^2/2m \qquad g > 0. \tag{3.41}$$

Then for a dimension $d = 3$, we may have a coexistence of two kinds of condensation [12]:

- The stability domain

$$Q^{BZ} \equiv \left\{(\beta > 0, \alpha \in \mathbb{R}) : \lim_\Lambda p_\Lambda^{BZ}(\beta, \alpha) \equiv \lim_\Lambda \left\{\frac{1}{\beta V} \ln \operatorname{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^{BZ} - \alpha N_\Lambda)}\right)\right\} < +\infty\right\}$$

  of H_Λ^{BZ} (3.40) is

$$Q^{BZ} = \{\beta > 0\} \times \{\alpha \leqslant \alpha_{\sup} = 0 < +\infty\}.$$

- There is $\alpha_{01} < 0$ such that there is *no condensation* for $\alpha < \alpha_{01}$.
- A macroscopic occupation of the mode $k = 0$ starts from $\alpha_{01} = \varepsilon_0 < 0$ to $\alpha_{02} = 0$, i.e.

$$\rho_0^{BZ}(\beta, \alpha) \equiv \lim_\Lambda \frac{1}{V} \langle a_0^* a_0 \rangle_{H_\Lambda^{BZ}}(\beta, \alpha) = \sup\left\{0, \frac{\alpha - \varepsilon_0}{2g_0}\right\}. \tag{3.42}$$

  This condensation (3.42) for $\alpha \in (\alpha_{01}, 0]$ is due to the instability implied by the diagonal interaction in the zero-mode ($\varepsilon_0 < 0$, $g_0 > 0$ in (3.40)), i.e. it is a *non-conventional* Bose condensation.
- Since $\rho_0^{BZ}(\beta, \alpha)$ (3.42) and the corresponding particle density

$$\rho^{BZ}(\beta, \alpha) \equiv \lim_\Lambda \frac{1}{V} \langle N_\Lambda \rangle_{H_\Lambda^{BZ}}(\beta, \alpha)$$

  attain their maxima at $\alpha = 0$, i.e.

$$\rho_0^{BZ}(\beta, 0) = -\frac{\varepsilon_0}{2g_0} \qquad \rho^{BZ}(\beta, 0) = \sup_{\alpha\leqslant 0} \rho^{BZ}(\beta, \alpha) < +\infty \tag{3.43}$$

  then a *conventional* BE condensation occurs for fixed particle densities $\rho > \rho^{BZ}(\beta, 0)$, but in a generalized sense, i.e. in modes close to $k = 0$. In fact since $g > 0$ (3.41) this second *conventional* BE condensation is *non-extensive* (type III),

$$\tilde{\rho}_0^{BZ}(\beta, \rho) \equiv \lim_{\delta\to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k\in\Lambda^* : 0 < \|k\| < \delta\}} \langle N_k \rangle_{H_\Lambda^{BZ}} = \sup\{0, \rho - \rho^{BZ}(\beta, 0)\} \tag{3.44}$$

  and *coexists* with the *non-conventional* one $\rho_0^{BZ}(\beta, 0)$ (3.42) for $\rho > \rho^{BZ}(\beta, 0)$. Then the 'global' Bose condensation is equal to

$$\tilde{\rho}_0^{BZ}(\beta, \rho) + \rho_0^{BZ}(\beta, 0) \qquad \text{for} \quad \rho > \rho^{BZ}(\beta, 0). \tag{3.45}$$

Another example of such a kind of thermodynamic behaviour is given by the Bogoliubov weakly imperfect Bose gas (cf equation (3.81) in [18]), see [10, 19, 20].

Therefore, in order to rigorously prove the existence of the same kinds of condensation for the models X (1.17) and SX (1.18), *a priori*, we should really take into account *in a separate way* the probable existence of six kinds of Bose condensation, cf appendix A. However in this paper, we restrict our discussion only to very simple cases. Note that for the moment, the non-conventional condensations explicitly found are only of type I [10, 12, 13, 19, 20].

By analogy with the previous example presented in [12], cf (3.40)–(3.45), we analyse the simple case in which a Bose condensation of type I (for simplicity in the zero-mode) exists for the model X (1.17), using a *fixed* inverse temperature β, on an interval of chemical potential α:

$$\lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H_\Lambda^X}(\beta, \alpha) > 0 \qquad \text{for} \quad \alpha \in [\alpha_{01}, \alpha_{02}] \quad \alpha_{01} < \alpha_{02} < \alpha_{\sup}. \tag{3.46}$$

We define by $\widetilde{\mu}_\beta(\alpha) \leqslant \mu_c(\beta)$ (2.25) the unique solution of equation

$$\widetilde{\alpha}_\beta(\widetilde{\mu}_\beta(\alpha)) = \alpha \leqslant \alpha_{\sup} \tag{3.47}$$

see [1]. If we consider the existence of $\alpha_{1,\beta}$ (condition 2.1 (iv)), then we assume that $\alpha_{1,\beta} \notin [\alpha_{01}, \alpha_{02}]$, i.e.

$$[\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)] \cap [\widetilde{\mu}_\beta(\alpha_{01}), \widetilde{\mu}_\beta(\alpha_{02})] = \{\varnothing\}$$

cf (2.24) and $\widetilde{\mu}_\beta(\alpha_{02}) < \mu_c(\beta)$ (2.25). Then one gets the following theorem.

Theorem 3.6. *If the non-superstable Hamiltonian H_Λ^X (1.17) verifies conditions 2.1, then one has*

$$\lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H_\Lambda^{SX}}(\beta, \mu) = \lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H_\Lambda^X}(\beta, \widetilde{\alpha}_\beta(\mu)) \tag{3.48}$$

for $\mu \notin (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$ (2.24) and $\mu < \mu_c(\beta)$ (2.25) with $\widetilde{\alpha}_\beta(\mu) < \alpha_{\sup}$ defined as the unique solution of (2.19). In fact

$$\lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H_\Lambda^{SX}}(\beta, \mu) = \lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H_\Lambda^X}(\beta, \widetilde{\alpha}_\beta(\mu)) > 0 \tag{3.49}$$

for

$$\mu \in D_\mu^{SX} \equiv \{\mu \in \mathbb{R} : \widetilde{\alpha}_\beta(\mu) \in [\alpha_{01}, \alpha_{02}]\} = [\widetilde{\mu}_\beta(\alpha_{01}), \widetilde{\mu}_\beta(\alpha_{02})] \neq \{\varnothing\}. \tag{3.50}$$

Proof.

(1) Again, in order to control the Bose condensation in the superstable model SX (1.18), we introduce the auxiliary Hamiltonians

$$H_{\Lambda,\gamma_0}^X \equiv H_\Lambda^X - \gamma_0 N_0 \qquad H_{\Lambda,\gamma_0}^{SX} \equiv H_\Lambda^{SX} - \gamma_0 N_0 = H_{\Lambda,\gamma_0}^X + \frac{\lambda}{V} N_\Lambda^2 \tag{3.51}$$

for $\delta > 0$ and $\lambda > 0$. We set

$$\begin{aligned} p_\Lambda^X(\beta, \alpha, \gamma_0) &\equiv \frac{1}{\beta V} \ln \operatorname{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_{\Lambda,\gamma_0}^X - \alpha N_\Lambda)}\right) \\ p_\Lambda^{SX}(\beta, \mu, \gamma_0) &\equiv \frac{1}{\beta V} \ln \operatorname{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_{\Lambda,\gamma_0}^{SX} - \mu N_\Lambda)}\right). \end{aligned} \tag{3.52}$$

The stability domain

$$Q_{\gamma_0}^X \equiv \left\{(\beta > 0, \alpha \in \mathbb{R}) : \lim_{\Lambda} p_\Lambda^X(\beta, \alpha, \gamma_0) < +\infty\right\} \neq \{\varnothing\}$$

associated with H_{Λ,γ_0}^X (3.51) is

$$Q_{\gamma_0}^X = \{\beta > 0\} \times \{\alpha < \alpha_{\sup}(\gamma_0) < +\infty\}. \tag{3.53}$$

Note that $\alpha_{\sup}(\gamma_0) \in [\alpha_{\sup} - \gamma_0, \alpha_{\sup}]$ if $\gamma_0 > 0$.

(2) Let $\mu \notin (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$ (2.24) and $\mu < \mu_c(\beta)$ (2.25), i.e. $\widetilde{\alpha}_\beta(\mu) \neq \alpha_{1,\beta}$ and $\widetilde{\alpha}_\beta(\mu) < \alpha_{\sup}$. Since the *non-superstable* Hamiltonian H_Λ^X (1.17) verifies conditions 2.1 for any $\alpha + \gamma_0 < \alpha_{\sup}$ ($\alpha < \alpha_{\sup}$), the pressure $p_\Lambda^X(\beta, \alpha, \gamma_0)$ (3.52) exists for any $\gamma_0 < b_0 \equiv (\alpha_{\sup} - \alpha) > 0$. Moreover, since $\gamma_0 < b_0$, following the same kinds of argument as in section 3.1, cf (3.9)–(3.15), the Hamiltonian $H_{\Lambda,\gamma}^X$ (3.9) also verifies (i) and (iii) of conditions 2.1. Then, via (2.19) one obtains

$$p^{SX}(\beta, \mu, \gamma_0) \equiv \lim_\Lambda p_\Lambda^{SX}(\beta, \mu, \gamma_0) = \inf_{\alpha < \alpha_{\sup}(\gamma_0)} \left\{ p^X(\beta, \alpha, \gamma_0) + \frac{(\mu - \alpha)^2}{4\lambda} \right\} \tag{3.54}$$

for $\mu \notin (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$ (2.24), $\mu < \mu_c(\beta)$ (2.25) and $\gamma_0 < b_0$, with

$$p^X(\beta, \alpha, \gamma_0) \equiv \lim_\Lambda p_\Lambda^X(\beta, \alpha, \gamma_0)$$

see (3.52).

(3) Note that $\left\{p_\Lambda^X(\beta, \alpha, \gamma_0)\right\}_\Lambda$ and $\left\{p_\Lambda^{SX}(\beta, \mu, \gamma_0)\right\}_\Lambda$ are also two sets of convex functions such that

$$\frac{1}{V}\langle N_0\rangle_{H_{\Lambda,\gamma_0}^X}(\beta, \alpha) = \partial_{\gamma_0} p_\Lambda^X(\beta, \alpha, \gamma_0)$$

$$\frac{1}{V}\langle N_0\rangle_{H_{\Lambda,\gamma_0}^{SX}}(\beta, \mu) = \partial_{\gamma_0} p_\Lambda^{SX}(\beta, \mu, \gamma_0).$$

Consequently following the same kinds of argument as for the proof of theorem 3.2 (cf (3.22)–(3.24) and (3.35)–(3.36)), by the Griffiths lemma [15, 16] and (3.54), for $\gamma_0 = 0$ and for $\mu \notin (\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta))$ (2.24) and $\mu < \mu_c(\beta)$ (2.25) we find

$$\lim_\Lambda \frac{1}{V}\langle N_0\rangle_{H_\Lambda^{SX}}(\beta, \mu) = \lim_\Lambda \frac{1}{V}\langle N_0\rangle_{H_\Lambda^X}(\beta, \widetilde{\alpha}_\beta(\mu))$$

with $\widetilde{\alpha}_\beta(\mu) < \alpha_{\sup}$ ($\widetilde{\alpha}_\beta(\mu) \neq \alpha_{1,\beta}$) defined as the unique solution of (2.19).

(4) Then, since $\alpha_{1,\beta} \notin [\alpha_{01}, \alpha_{02}]$ and $\alpha_{02} < \alpha_{\sup}$, via (3.46) we have (3.49) for $\mu \in D_\mu^{SX} = [\widetilde{\mu}_\beta(\alpha_{01}), \widetilde{\mu}_\beta(\alpha_{02})] \neq \{\varnothing\}$ (3.50), see (3.47). □

For high fixed particle densities $\rho > 0$, if the non-superstable Hamiltonian H_Λ^X (1.17) verifies conditions 2.1, the results of corollary 3.5 and theorem 3.6 mean as follows:

Corollary 3.7. *If (3.46) is verified then*

$$\lim_\Lambda \frac{1}{V}\langle N_0\rangle_{H_\Lambda^{SX}}(\beta, \mu_\Lambda(\rho)) = \lim_\Lambda \frac{1}{V}\langle N_0\rangle_{H_\Lambda^X}(\beta, \alpha_\Lambda(\rho)) > 0 \tag{3.55}$$

for

$$\rho \in D_\rho^{SX} = [\rho_{01}, \rho_{02}] \in \left(0, \rho_{\inf}^X(\beta, \alpha_{1,\beta})\right) \cup \left(\rho_{\sup}^X(\beta, \alpha_{1,\beta}), \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha)\right) \neq \{\varnothing\} \tag{3.56}$$

with $\alpha_\Lambda(\rho)$ and $\mu_\Lambda(\rho)$ defined by (3.2) and (3.7) respectively. Here ρ_{01} and ρ_{02} are defined by

$$\begin{aligned}\rho_{01} &\equiv \rho^X(\beta, \alpha_{01}) = \rho^{SX}(\beta, \widetilde{\mu}_\beta(\alpha_{01})) \quad \Leftrightarrow \quad \mu(\rho_{01}) \\ &= \widetilde{\mu}_\beta(\alpha_{01}) \quad \Leftrightarrow \quad \widetilde{\alpha}_\beta(\mu(\rho_{01})) = \alpha_{01} < \alpha_{\sup} \\ \rho_{02} &\equiv \rho^X(\beta, \alpha_{02}) = \rho^{SX}(\beta, \widetilde{\mu}_\beta(\alpha_{02})) \quad \Leftrightarrow \quad \mu(\rho_{02}) \\ &= \widetilde{\mu}_\beta(\alpha_{02}) \quad \Leftrightarrow \quad \widetilde{\alpha}_\beta(\mu(\rho_{02})) = \alpha_{02} < \alpha_{\sup}\end{aligned} \tag{3.57}$$

see (3.47).

Moreover, if the Bose condensation (3.46) coexists only with a conventional BE condensation of any type I, II or III for the model X (1.17) for $\rho > \rho^X(\beta, \alpha_{\sup})$ *(2.17), then, assuming the conditions of theorem 3.6 and that there is* $\delta > 0$ *such that* $\alpha_{\sup}(\gamma_0) = \alpha_{\sup}$ *for* $\gamma_0 \in [-\delta, \delta]$, *cf (3.53), the condensation (3.55) coexists also for* $\rho > \rho^X(\beta, \alpha_{\sup})$ *with a conventional BE condensation for the model SX (1.18).*

Proof.

(1) $\rho < \lim_{\alpha \to \alpha_{\sup}^-} \rho^X(\beta, \alpha)$, i.e. $\mu(\rho) < \mu_c(\beta)$ (2.25) where $\mu(\rho)$ is defined by (2.29), see also (3.8). Considering the existence of $\alpha_{1,\beta}$ (condition 2.1 (iv)), since $\alpha_{1,\beta} \notin [\alpha_{01}, \alpha_{02}], [\rho_{01}, \rho_{02}] \notin \left[\rho^X_{\inf}(\beta, \alpha_{1,\beta}), \rho^X_{\sup}(\beta, \alpha_{1,\beta})\right]$ (2.15), i.e. $\mu(\rho) \notin [\mu_{1,\inf}(\beta), \mu_{1,\sup}(\beta)]$ (2.24). Then (3.55)–(3.56) are only a consequence of (2.30) and (3.47)–(3.49). Note that the different equations of (3.57) are also a consequence of (2.20)–(2.25).

(2) If there exists $\delta > 0$ such that $\alpha_{\sup}(\gamma_0) = \alpha_{\sup}$ for $\gamma_0 \in [-\delta, \delta]$, cf (3.53), then following the same kind of argument derived from (3.22)–(3.23), (3.25)–(3.29) and (3.35)–(3.36), we can extend (3.48) to find

$$\lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H^{SX}_\Lambda}(\beta, \mu_\Lambda(\rho)) = \lim_{\Lambda} \frac{1}{V} \langle N_0 \rangle_{H^X_\Lambda}(\beta, \alpha_\Lambda(\rho)) \tag{3.58}$$

for $\rho \geqslant \rho^X(\beta, \alpha_{\sup})$ (2.17). Therefore, the last statement of this corollary comes from corollary 3.5 combined with (3.58). □

In fact the results of theorem 3.6 and corollary 3.7 could be *extended* to any Bose condensation of type I, II or III which may exist for the model X (1.17) on an interval of chemical potential $\alpha \in [\alpha_{01}, \alpha_{02}]$, $\alpha_{01} < \alpha_{02} \leqslant \alpha_{\sup}$.

Note that the second conventional BE condensation, which may exist for the model X (1.17), persists in the superstabilized model SX (1.18) but we do not know *a priori* if its type changes or not. In fact, we conjecture the following statement:

Conjecture 3.8. *If some sufficient conditions such as conditions 2.1 are verified, the models X (1.17) and SX (1.18) manifest exactly the same kinds of condensation, type included.*

Conjecture 3.8 is of course verified by the PBG (1.2) and the IBG (1.1) (cf [5, 6, 9]), but also by the Hamiltonians H^{BZ}_Λ(3.40) and

$$H^{SBZ}_\Lambda \equiv H^{BZ}_\Lambda + \frac{\lambda}{V} N^2_\Lambda \qquad \lambda > 0$$

see [12, 13]. In fact, note that the model BZ (3.40) verifies all the assumptions of corollary 3.7, see (3.40)–(3.45) [12].

4. Canonical ensemble versus grand-canonical ensemble

In this section, we only need that the non-superstable Hamiltonian H^X_Λ (1.17) verifies the three assumptions (i)–(iii) of conditions 2.1.

For the moment, we have presented a method to superstabilize some Hamiltonian H^X_Λ (1.17). Considering the last section, for a fixed particle density ρ, the models X and SX seem to have exactly the same thermodynamic behaviour, at least on the level of Bose condensations, see corollaries 3.5 and 3.7, conjecture 3.8. To go further in the understanding of relations between the models X and SX we analyse now their corresponding (infinite volume) Gibbs states:

- in the *canonical* ensemble (β, ρ)

$$\omega^{X}_{\beta,\rho}(-) \equiv \lim_{\Lambda}\langle - \rangle_{H^{X}_{\Lambda}}(\beta, \rho) \qquad \omega^{SX}_{\beta,\rho}(-) \equiv \lim_{\Lambda}\langle - \rangle_{H^{SX}_{\Lambda}}(\beta, \rho) \tag{4.1}$$

formally the (*infinite volume*) canonical Gibbs states corresponding to the Bose gases X (1.17) and SX (1.18) respectively;

- in the *grand-canonical* ensembles, (β, α) for the model X (1.17) and (β, μ) for the model SX (1.18), i.e. formally

$$\omega^{X}_{\beta,\alpha}(-) \equiv \lim_{\Lambda}\langle - \rangle_{H^{X}_{\Lambda}}(\beta, \alpha) \qquad \omega^{SX}_{\beta,\mu}(-) \equiv \lim_{\Lambda}\langle - \rangle_{H^{SX}_{\Lambda}}(\beta, \mu). \tag{4.2}$$

We recall that $\langle - \rangle_{H^{X}_{\Lambda}}(\beta, \rho)$ and $\langle - \rangle_{H^{SX}_{\Lambda}}(\beta, \rho)$ are the (*finite volume*) canonical Gibbs states associated with H^{X}_{Λ} (1.17) and H^{SX}_{Λ} (1.18) respectively, whereas $\langle - \rangle_{H^{X}_{\Lambda}}(\beta, \alpha)$ and $\langle - \rangle_{H^{SX}_{\Lambda}}(\beta, \mu)$ are the corresponding (*finite volume*) grand-canonical Gibbs states, see (2.6).

In the *canonical ensemble*, note that the two models X and SX are completely equivalent on the level of Gibbs states. Indeed, via (1.18), (2.1) and (2.3), for any operator A in the set of operators acting on $\mathcal{F}^{B}_{\Lambda}$ (1.4) such that

$$\langle A \rangle_{H^{X}_{\Lambda}}(\beta, \rho) < +\infty \qquad \text{or} \qquad \langle A \rangle_{H^{SX}_{\Lambda}}(\beta, \rho) < +\infty$$

one has

$$\mathrm{Tr}_{\mathcal{H}^{(n)}_{B}}\left(\left\{A\,\mathrm{e}^{-\beta H^{SX}_{\Lambda}}\right\}^{(n)}\right) = \mathrm{e}^{-\beta\lambda\rho^{2}}\,\mathrm{Tr}_{\mathcal{H}^{(n)}_{B}}\left(\left\{A\,\mathrm{e}^{-\beta H^{X}_{\Lambda}}\right\}^{(n)}\right) \tag{4.3}$$

which directly implies

$$\langle A \rangle_{H^{SX}_{\Lambda}}(\beta, \rho) = \langle A \rangle_{H^{X}_{\Lambda}}(\beta, \rho) \tag{4.4}$$

for $\beta > 0$ and $\rho > 0$, see (2.6). In fact, in the thermodynamic limit, formally we get

$$\omega^{X}_{\beta,\rho}(-) = \omega^{SX}_{\beta,\rho}(-). \tag{4.5}$$

4.1. Strong equivalence between canonical and grand-canonical ensembles

The notion of strong equivalence between the canonical and the grand-canonical ensembles means in fact that, in term of Gibbs states, fixing the particle density ρ in the *grand-canonical* ensemble corresponds to analysing the infinite volume SX Gibbs state (4.1) in the *canonical* ensemble. This strong equivalence between the two corresponding ensembles may not be verified for the model X, whereas the superstable system SX should verify it. In fact, for superstable interaction the large deviation principle was established by Georgii [2] in 1994. More precisely, the corresponding paper [2] shows, for superstable gases, the asymptotic equivalence of microcanonical and grand-canonical Gibbs distributions and deduces a variational expression for the thermodynamic entropy density. Note also that Lewis *et al* prove in [21–26] the strong equivalence of ensembles (canonical/grand-canonical) for very general state spaces, discrete or continuous, compact or non-compact, but with bounded interactions. The authors were indeed interested in getting a large deviation principle for the empirical measure for the so-called tau-topology. To resume, the main restriction of this work [21–26] is the fact that the energy is a sum of local bounded observables. An application is done in [27] for lattice systems.

Here, the aim of this section is *only* to explain this notion of strong equivalence in our (*specific*) superstable model SX (1.18). Though the model SX combined with (4.4), actually we are interested in using this strong equivalence to give a new way to study the Bose gas X in the canonical ensemble. But first, assuming as verified the assumptions (i)–(iii) of conditions 2.1, we give some results which are a simple application of [2] to the model SX.

Let us consider by A_Λ a (positive) quasi-local operator acting on

$$\mathcal{F}_\Lambda^B \subset \mathcal{F}_\infty^B \equiv \bigoplus_{n=0}^{+\infty} (L^2(\mathbb{R}^{nd}))_{\text{symm}} \tag{4.6}$$

such that

$$\lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^X}(\beta, \rho) = \lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^{SX}}(\beta, \rho) < +\infty \tag{4.7}$$

for any $\beta > 0$ and $\rho > 0$ whereas for $\alpha < \alpha_{\text{sup}}$ or $\mu \in \mathbb{R}$

$$\lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^X}(\beta, \alpha) < +\infty \qquad \lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^{SX}}(\beta, \mu) < +\infty. \tag{4.8}$$

Then, one gets the following theorem:

Theorem 4.1. *We have*

$$\lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^{SX}}(\beta, \mu) = \lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^{SX}}(\beta, \rho = \rho^{SX}(\beta, \mu)) = \lim_\Lambda \langle A_\Lambda \rangle_{H_\Lambda^X}(\beta, \rho = \rho^{SX}(\beta, \mu)) \tag{4.9}$$

with $\rho^{SX}(\beta, \mu)$ defined by (2.20).

Proof. The proof is mostly a direct consequence of [2]. Here, we only give a simple but instructive proof considering our specific model SX.

Using the free-energy density $f_\Lambda^X(\beta, \rho)$ (2.2) combined with (1.18), (2.1) and (2.6), one has

$$\langle A_\Lambda \rangle_{H_\Lambda^{SX}}(\beta, \mu) = \frac{1}{V} \sum_{t \in \{0, 1/V, 2/V, \ldots, +\infty\}} \nu_\Lambda(t) h_\Lambda(t) \tag{4.10}$$

with

$$\begin{aligned} &h_\Lambda(t \neq 0) \equiv \langle A_\Lambda \rangle_{H_\Lambda^X}(\beta, t) \geqslant 0 \qquad h_\Lambda(0) \equiv \lim_{t \to 0^+} \langle A_\Lambda \rangle_{H_\Lambda^X}(\beta, t) \\ &\nu_\Lambda(t) \equiv \frac{\exp(\beta V\{\mu t - \lambda t^2 - f_\Lambda^X(\beta, t)\})}{\frac{1}{V}\sum_{t \in \{0, 1/V, 2/V, \ldots, +\infty\}} \exp\left(\beta V\left\{\mu t - \lambda t^2 - f_\Lambda^X(\beta, t)\right\}\right)} \end{aligned} \tag{4.11}$$

with $f_\Lambda^X(\beta, 0) \equiv \lim_{t \to 0^+} f_\Lambda^X(\beta, t)$. Note that

$$p_\Lambda^{SX}(\beta, \mu) = \frac{1}{\beta V} \ln \sum_{n=0}^{+\infty} \exp\left(\beta V\left\{\mu\left(\frac{n}{V}\right) - \lambda\left(\frac{n}{V}\right)^2 - f_\Lambda^X\left(\beta, \frac{n}{V}\right)\right\}\right)$$

cf (2.2) and (2.4). $p^{SX}(\beta, \mu)$ (2.19) is the Legendre transformation of $f^{SX}(\beta, \rho)$ (2.18), i.e.

$$p^{SX}(\beta, \mu) = \sup_{t > 0}\{\mu t - f^{SX}(\beta, t)\} = \sup_{t > 0}\{\mu t - \lambda t^2 - f^X(\beta, t)\}. \tag{4.12}$$

Then, since the particle density $\rho^{SX}(\beta, \mu)$ (2.20) is a strictly increasing function for $\mu \in \mathbb{R}$ [1] which via (4.12) verifies

$$\rho^{SX}(\beta, \mu) = \partial_\mu p^{SX}(\beta, \mu) = \partial_\mu \left(\sup_{t > 0}\{\mu t - \lambda t^2 - f^X(\beta, t)\}\right)$$

we deduce that $\rho^{SX}(\beta, \mu) = \rho$ is the *unique* solution of

$$\sup_{t > 0}\{\mu t - \lambda t^2 - f^X(\beta, t)\} = \mu\rho - \lambda\rho^2 - f^X(\beta, \rho). \tag{4.13}$$

Also, there are $M > 0$ and $K < \alpha_{\text{sup}}$ such that

$$\mu t - \lambda t^2 - f_\Lambda^X(\beta, t) < Kt \tag{4.14}$$

for $t \geqslant M$. Then, for each $\varepsilon > 0$,

$$\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}} h_\Lambda(t)\nu_\Lambda(t) = \exp\left(-\beta V\left\{p_\Lambda^{SX}(\beta,\mu) - \frac{1}{\beta V}\ln V\right\}\right)$$
$$\times h_\Lambda(\rho_{\varepsilon,\Lambda})\left\{\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap[\rho-\varepsilon,\rho+\varepsilon]}\nu_\Lambda(t)\right\} + \exp\left(-\beta V\left\{p_\Lambda^{SX}(\beta,\mu)\right.\right.$$
$$\left.\left.- \frac{1}{\beta V}\ln V\right\}\right) h_\Lambda(\rho_{\varepsilon,\Lambda}^-)\left\{\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap[0,\rho-\varepsilon)}\nu_\Lambda(t)\right\}$$
$$+ \exp\left(-\beta V\left\{p_\Lambda^{SX}(\beta,\mu) - \frac{1}{\beta V}\ln V\right\}\right)$$
$$\times h_\Lambda\left(\rho_{\varepsilon,\Lambda}^+\right)\left\{\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap(\rho+\varepsilon,M)}\nu_\Lambda(t)\right\} + \exp\left(-\beta V p_\Lambda^{SX}(\beta,\mu)\right)$$
$$\times\left\{\sum_{n\geqslant M\times V}^{+\infty}\exp\left(\beta V\left\{\mu\left(\frac{n}{V}\right) - \lambda\left(\frac{n}{V}\right)^2 - f_\Lambda^X\left(\beta,\frac{n}{V}\right)\right\}\right)h_\Lambda\left(\frac{n}{V}\right)\right\} \quad (4.15)$$

where $\rho_{\varepsilon,\Lambda} \in [\rho-\varepsilon,\rho+\varepsilon]$, $\rho_{\varepsilon,\Lambda}^- \in [0,\rho-\varepsilon)$, and $\rho_{\varepsilon,\Lambda}^+ \in (\rho+\varepsilon, M)$:

$$h_\Lambda(\rho_{\varepsilon,\Lambda}) = \frac{1}{2\varepsilon}\left\{\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap[\rho-\varepsilon,\rho+\varepsilon]} h_\Lambda(t)\nu_\Lambda(t)\right\}$$
$$h_\Lambda(\rho_{\varepsilon,\Lambda}^-) = \frac{1}{\rho-\varepsilon}\left\{\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap[0,\rho-\varepsilon)} h_\Lambda(t)\nu_\Lambda(t)\right\}$$
$$h_\Lambda\left(\rho_{\varepsilon,\Lambda}^+\right) = \frac{1}{M-(\rho+\varepsilon)}\left\{\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap(\rho+\varepsilon,M)} h_\Lambda(t)\nu_\Lambda(t)\right\}.$$

Using the large deviation principle [21, 23],

$$\lim_\Lambda \frac{1}{\beta V}\ln\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap(a,b)}\nu_\Lambda(t) - p_\Lambda^{SX}(\beta,\mu)$$
$$= \sup_{t\in(a,b)}\{\mu t - \lambda t^2 - f^X(\beta,t)\} - \sup_{t>0}\{\mu t - \lambda t^2 - f^X(\beta,t)\}$$

for any $a, b \in \mathbb{R}^+$ and thus

$$\lim_\Lambda\left\{\exp\left(-\beta V\left\{p_\Lambda^{SX}(\beta,\mu) - \frac{1}{\beta V}\ln V\right\}\right)\frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}\cap(a,b)}\nu_\Lambda(t)\,\mathrm{d}t\right\}$$
$$= \chi_{(a,b)}(t=\rho) \quad (4.16)$$

where $\chi_{(a,b)}(t)$ is a characteristic function of (a,b). By (4.7) the function $h_\Lambda(t)$ (4.11) is well defined in the thermodynamic limit for any $t > 0$. Therefore, since by (4.14) one has

$$\sum_{n\geqslant M\times V}^{+\infty} \exp\left(\beta V\left\{\mu\left(\frac{n}{V}\right)-\lambda\left(\frac{n}{V}\right)^2 - f_\Lambda^X\left(\beta,\frac{n}{V}\right)\right\}\right) h_\Lambda\left(\frac{n}{V}\right)$$
$$\leqslant \sum_{n\geqslant M\times V}^{+\infty} \mathrm{e}^{\beta K n} h_\Lambda\left(\frac{n}{V}\right) < +\infty$$

with $K < \alpha_{\text{sup}}$ (cf (4.8)), we deduce from (4.15) and (4.16) that

$$\lim_{\varepsilon\to 0^+}\lim_\Lambda \frac{1}{V}\sum_{t\in\{0,1/V,2/V,\ldots,+\infty\}} h_\Lambda(t)\nu_\Lambda(t) = \lim_{\varepsilon\to 0^+}\lim_\Lambda h_\Lambda(\rho_{\varepsilon,\Lambda}) = \lim_\Lambda \langle A_\Lambda\rangle_{H_\Lambda^{SX}}(\beta,\rho^{SX}(\beta,\mu))$$

see (4.10)–(4.11), i.e. combined with (4.4) one has (4.9). □

Corollary 4.2. *For $\beta > 0$ and $\rho > 0$, we deduce from theorem 4.1 that*

$$\lim_\Lambda \langle A_\Lambda\rangle_{H_\Lambda^{SX}}(\beta,\mu_\Lambda(\rho)) = \lim_\Lambda \langle A_\Lambda\rangle_{H_\Lambda^{SX}}(\beta,\rho) = \lim_\Lambda \langle A_\Lambda\rangle_{H_\Lambda^{X}}(\beta,\rho)$$

with $\mu_\Lambda(\rho)$ defined by (3.7).

In fact, thermodynamic behaviour of the *superstable* model SX (1.18) in the *canonical* ensemble is '*identical*' to that in the *grand-canonical* ensemble and then, the analysis of the X Gibbs state (4.1) in the *canonical* ensemble could be done in the thermodynamic limit by using the *grand-canonical SX* Gibbs state (4.2) for a fixed particle density, see corollary 4.2.

4.2. Generating functionals

A first application of (4.4), theorem 4.1 and corollary 4.2 is the study of the generating functionals of Gibbs states (4.1) and (4.2). Indeed originally initiated by Araki and Woods [28], see also [29–32] or appendix B, the description of Gibbs states can be given by using the representations of the CCR (canonical commutation relations).

Hence, following these works [28–32], using the Fock representation of the CCR [32] over the space $\mathcal{D}_\Lambda$ of $C_0^\infty(\Lambda)$-functions with compact support contained in Λ, we define by $\mathbb{E}_{\Lambda,c}^X(\beta,\rho;h)$ and $\mathbb{E}_{\Lambda,c}^{SX}(\beta,\rho;h)$ the *canonical* generating functionals corresponding to H_Λ^X (1.17) and H_Λ^{SX} (1.18) respectively:

$$\mathbb{E}_{\Lambda,c}^X(\beta,\rho;h) \equiv \left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{H_\Lambda^X}(\beta,\rho) \qquad \mathbb{E}_{\Lambda,c}^{SX}(\beta,\rho;h) \equiv \left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{H_\Lambda^{SX}}(\beta,\rho). \tag{4.17}$$

Let us also consider by

$$\mathbb{E}_{\Lambda}^X(\beta,\alpha;h) \equiv \left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{H_\Lambda^X}(\beta,\alpha) \qquad \mathbb{E}_{\Lambda}^{SX}(\beta,\mu;h) \equiv \left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{H_\Lambda^{SX}}(\beta,\mu) \tag{4.18}$$

the *grand-canonical* generating functionals corresponding to H_Λ^X (1.17) and H_Λ^{SX} (1.18) respectively. Then assuming again that the Hamiltonian H_Λ^X verifies (i)–(iii) of conditions 2.1, via (4.4) and theorem 4.1 for $A_\Lambda = W^{\mathcal{F}_\Lambda^B}(h)$, one gets the following result:

Corollary 4.3. *In the canonical ensemble,*

$$\mathbb{E}_{\Lambda,c}^{SX}(\beta,\rho;h) = \mathbb{E}_{\Lambda,c}^{X}(\beta,\rho;h) \tag{4.19}$$

for $\beta > 0$ and $\rho > 0$, whereas in the grand-canonical ensemble,

$$\mathbb{E}^{SX}(\beta,\mu;h) \equiv \lim_\Lambda \mathbb{E}_\Lambda^{SX}(\beta,\mu;h) = \mathbb{E}_c^{SX}(\beta,\rho^{SX}(\beta,\mu);h) = \mathbb{E}_c^{X}(\beta,\rho^{SX}(\beta,\mu);h) \tag{4.20}$$

for $(\beta,\mu)\in Q^S\equiv\{\beta>0\}\times\{\mu\in\mathbb{R}\}$ *and* h *in the space* $\mathcal{D}=\bigcup_{\Lambda\subset\mathbb{R}^d}\mathcal{D}_\Lambda$ *of* C^∞*-smooth functions on* $\mathbb{R}^d$ *having compact support. Here*

$$\mathbb{E}_c^X(\beta,\rho;h)\equiv\lim_\Lambda\mathbb{E}_{\Lambda,c}^X(\beta,\rho;h)\qquad\mathbb{E}_c^{SX}(\beta,\rho;h)\equiv\lim_\Lambda\mathbb{E}_{\Lambda,c}^{SX}(\beta,\rho;h).$$

In particular, considering the particle density ρ as a parameter, by corollary 4.2 one has

$$\mathbb{E}_c^X(\beta,\rho;h)=\mathbb{E}_c^{SX}(\beta,\rho;h)=\mathbb{E}^{SX}(\beta,\mu(\rho);h)=\lim_\Lambda\mathbb{E}_\Lambda^{SX}(\beta,\mu_\Lambda(\rho);h)\tag{4.21}$$

with $\mu_\Lambda(\rho)$ solution of (3.7) and with the corresponding thermodynamic limit $\mu(\rho)$ defined by (2.29).

A direct application of corollary 4.3 or (4.21) could be done by using the PBG (1.2) example for which the corresponding superstable model is the IBG (1.1). Thus from (4.21) the *grand-canonical* generating functional corresponding to H_Λ^{IBG} (1.1),

$$\mathbb{E}_\Lambda^{IBG}(\beta,\mu;h)\equiv\left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{H_\Lambda^{IBG}}(\beta,\mu)=\frac{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(W^{\mathcal{F}_\Lambda^B}(h)\,\mathrm{e}^{-\beta(H_\Lambda^{IBG}-\mu N_\Lambda)}\right)}{\mathrm{Tr}_{\mathcal{F}_\Lambda^B}\left(\mathrm{e}^{-\beta(H_\Lambda^{IBG}-\mu N_\Lambda)}\right)}\tag{4.22}$$

for $\mu=\mu_\Lambda^{IBG}(\rho)$ (4.24) gives in the thermodynamic limit

$$\mathbb{E}_c^{PBG}(\beta,\rho;h)=\mathbb{E}_c^{IBG}(\beta,\rho;h)=\mathbb{E}^{IBG}\left(\beta,\mu^{IBG}(\rho);h\right)\equiv\lim_\Lambda\mathbb{E}_\Lambda^{IBG}\left(\beta,\mu_\Lambda^{IBG}(\rho);h\right)\tag{4.23}$$

with $\mu_\Lambda^{IBG}(\rho)$ satisfying

$$\rho_\Lambda^{IBG}\left(\beta,\mu_\Lambda^{IBG}(\rho)\right)\equiv\left\langle\frac{N_\Lambda}{V}\right\rangle_{H_\Lambda^{IBG}}\left(\beta,\mu_\Lambda^{IBG}(\rho)\right)=\rho>0\tag{4.24}$$

and the corresponding thermodynamic limit $\mu^{IBG}(\rho)$ solution of (1.13). Here

$$\mathbb{E}_c^{PBG}(\beta,\rho;h)\equiv\lim_\Lambda\left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{T_\Lambda}(\beta,\rho)\qquad\mathbb{E}_c^{IBG}(\beta,\rho;h)\equiv\lim_\Lambda\left\langle W^{\mathcal{F}_\Lambda^B}(h)\right\rangle_{H_\Lambda^{IBG}}(\beta,\rho).$$

Using the results of [28–32], the canonical generating functional $\mathbb{E}_c^{PBG}(\beta,\rho;h)$ is equal to

$$\mathbb{E}_c^{PBG}(\beta,\rho;h)=\exp\left\{-\frac{1}{4}\|h\|^2-\frac{1}{2}\int_{\mathbb{R}^d}\frac{|h_k|^2}{\mathrm{e}^{\beta(\varepsilon_k-\alpha^{PBG}(\rho))}-1}\,\mathrm{d}^dk\right\}\tag{4.25}$$

for $\rho<\lim_{\alpha\to0^-}\rho^{PBG}(\beta,\alpha)$ (1.10) with $\alpha^{PBG}(\rho)$ verifying

$$\rho^{PBG}(\beta,\alpha^{PBG}(\rho))=\rho\qquad\text{for}\quad\rho<\lim_{\alpha\to0^-}\rho^{PBG}(\beta,\alpha).\tag{4.26}$$

Here

$$h_k\equiv(\mathrm{e}^{\mathrm{i}kx},h)_{L^2(\mathbb{R}^d)}\qquad\text{for}\quad k\in\mathbb{R}^d$$

is the Fourier decomposition of functions $h\in\mathcal{D}$.

For dimensions $d\geqslant3$, one has a saturation of the particle density $\rho^{PBG}(\beta,\alpha)$ (1.10), i.e. there is a critical density $\rho^{PBG}(\beta,0)$ (1.9). Then for $\rho\geqslant\rho^{PBG}(\beta,0)$ (1.9), we have

$$\mathbb{E}_c^{PBG}(\beta,\rho;h)=J_0\left(\sqrt{2(\rho-\rho^{PBG}(\beta,0))}|h_0|\right)\exp\left\{-\frac{1}{4}\|h\|^2-\frac{1}{2}\int_{\mathbb{R}^d}\frac{|h_k|^2}{\mathrm{e}^{\beta\varepsilon_k}-1}\,\mathrm{d}^dk\right\}\tag{4.27}$$

cf [28, 29].

Therefore, combining (4.25) and (4.27) with (4.23) we deduce

$$\begin{aligned}\mathbb{E}_c^{IBG}(\beta,\rho;h)&=\mathbb{E}_c^{PBG}(\beta,\rho;h)=\mathbb{E}^{IBG}(\beta,\mu^{IBG}(\rho);h)\\&=\exp\left\{-\frac{1}{4}\|h\|^2-\frac{1}{2}\int_{\mathbb{R}^d}\frac{|h_k|^2}{\mathrm{e}^{\beta(\varepsilon_k-\alpha^{PBG}(\rho))}-1}\,\mathrm{d}^dk\right\}\end{aligned}\tag{4.28}$$

for $\rho < \lim_{\alpha\to 0^-} \rho^{PBG}(\beta,\alpha)$, whereas if $d \geqslant 3$, for $\rho \geqslant \rho^{PBG}(\beta,0)$ (1.9) ($\mu^{IBG}(\rho) \geqslant \mu_c^{IBG}(\beta)$ (1.8)), one obtains

$$\begin{aligned}\mathbb{E}_c^{IBG}(\beta,\rho;h) &= \mathbb{E}_c^{PBG}(\beta,\rho;h) = \mathbb{E}^{IBG}(\beta,\mu^{IBG}(\rho);h)\\ &= J_0\big(\sqrt{2(\rho-\rho^{PBG}(\beta,0))}|h_0|\big)\exp\left\{-\frac{1}{4}\|h\|^2 - \frac{1}{2}\int_{\mathbb{R}^d}\frac{|h_k|^2}{\mathrm{e}^{\beta\varepsilon_k}-1}\,\mathrm{d}^d k\right\}.\end{aligned} \tag{4.29}$$

In fact for the corresponding IBG Gibbs state one can fix either the chemical potential μ which implies a particle density $\rho^{IBG}(\beta,\mu)$, or the particle density ρ which implies a chemical potential $\mu^{IBG}(\rho)$ (1.13). In particular, the analysis of the *grand-canonical* IBG Gibbs state with a fixed particle density ρ corresponds, in the thermodynamic limit, to studying the *canonical* IBG Gibbs state, see (4.28) and (4.29). This property is verified by any superstabilized model SX (1.18), see (4.19)–(4.21).

Note that this property, satisfied by the IBG, is not verified by the PBG for $\rho > \rho^{PBG}(\beta,0)$ (1.9) ($d \geqslant 3$), even if their Bose condensation phenomenon is similar (cf (1.15) and more generally section 3). Indeed the papers [30, 31] show that

$$\mathbb{E}^{PBG}(\beta,\alpha^{PBG}(\rho)\leqslant 0;h) \equiv \lim_{\Lambda}\mathbb{E}_\Lambda^{PBG}\left(\beta,\alpha_\Lambda^{PBG}(\rho);h\right) = \mathbb{E}_c^{PBG}(\beta,\rho;h)$$

for $\rho \leqslant \rho^{PBG}(\beta,0)$ (1.9) ($d \geqslant 3$), see (4.25), whereas for $\rho > \rho^{PBG}(\beta,0)$

$$\begin{aligned}\widetilde{\mathbb{E}}^{PBG}(\beta,\rho;h) &\equiv \lim_{\Lambda}\mathbb{E}_\Lambda^{PBG}\left(\beta,\alpha_\Lambda^{PBG}(\rho);h\right)\\ &= \exp\left\{-\frac{1}{2}|h_0|^2(\rho-\rho^{PBG}(\beta,0))\right\}\exp\left\{-\frac{1}{4}\|h\|^2-\frac{1}{2}\int_{\mathbb{R}^d}\frac{|h_k|^2}{\mathrm{e}^{\beta\varepsilon_k}-1}\,\mathrm{d}^d k\right\}.\end{aligned}$$

In fact, for $\rho > \rho^{PBG}(\beta,0)$

$$\widetilde{\widetilde{\mathbb{E}}}^{PBG}(\beta,\rho;h) = \int_{\rho^{PBG}(\beta,0)}^{+\infty}\mathbb{E}_c^{PBG}(\beta,t;h)\exp\left\{-\frac{t-\rho^{PBG}(\beta,0)}{[\rho-\rho^{PBG}(\beta,0)]}\right\}\frac{\mathrm{d}t}{[\rho-\rho^{PBG}(\beta,0)]}$$

cf [29], and then

$$\widetilde{\widetilde{\mathbb{E}}}^{PBG}(\beta,\rho;h) \neq \mathbb{E}_c^{PBG}(\beta,\rho;h) \qquad \rho > \rho^{PBG}(\beta,0) \tag{4.30}$$

see (4.27). Inequality (4.30) comes from the non-bijectivity for $\rho > \rho^{PBG}(\beta,0)$ ($d \geqslant 3$) of the function

$$\rho \to \alpha^{PBG}(\rho) = \lim_{\Lambda}\alpha_\Lambda^{PBG}(\rho) = \begin{cases}\alpha^{PBG}(\rho) < 0 & \text{for} \quad \rho < \rho^{PBG}(\beta,0) \text{ cf (2.27)}\\ \alpha^{PBG}(\rho) = 0 & \text{for} \quad \rho \geqslant \rho^{PBG}(\beta,0) \text{ cf (2.28)}\end{cases}$$

see (3.2) and (3.6). More generally, in the thermodynamic limit, the strong equivalence between the canonical ensemble (β,ρ) and the grand-canonical ensemble (β,α) for the *non-superstable* Bose gas X (1.17) is directly related to the function $\rho \to \alpha^X(\rho)$. As soon as the function $\rho \to \alpha^X(\rho)$ is bijective on a subdomain $\mathcal{R}_1$ of particle density $\rho > 0$, this strong equivalence of ensembles is verified for $\rho \in \mathcal{R}_1$. However, the existence of a subdomain $\mathcal{R}_2$ of $\rho > 0$ where the function $\rho \to \alpha^X(\rho)$ is *not injective* (remark 2.3) implies the *breakdown* of the strong equivalence of ensembles for $\rho \in \mathcal{R}_2$, see also discussions in [33].

4.3. Bose condensations in the canonical ensemble

To go further, note that the analysis of the superstable Bose gas SX (1.18) may give a lot of information about the thermodynamic behaviour of the non-superstable model X (1.17) in the *canonical* ensemble. For example, using (4.21) for a fixed particle density ρ, we deduce, in the thermodynamic limit, the *canonical* X generating functional $\mathbb{E}_c^X(\beta, \rho; h)$ from the *grand-canonical* SX generating functional $\mathbb{E}^{SX}(\beta, \mu(\rho); h)$.

Therefore, the end of this section just explains that from the knowledge of the existence of Bose condensations in the model SX (1.18) (cf section 3) we deduce the existence of the same Bose condensations for the model X (1.17) but in the *canonical* ensemble. Indeed via corollary 4.2 for

$$A_\Lambda = \frac{1}{V} \sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}} N_k \tag{4.31}$$

one has

$$\begin{aligned}\lim_{\delta\to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}} \langle N_k\rangle_{H_\Lambda^{SX}}(\beta, \mu_\Lambda(\rho)) &= \lim_{\delta\to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}} \langle N_k\rangle_{H_\Lambda^{SX}}(\beta, \rho) \\ &= \lim_{\delta\to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k\in\Lambda^*:0\leqslant\|k\|<\delta\}} \langle N_k\rangle_{H_\Lambda^{X}}(\beta, \rho)\end{aligned} \tag{4.32}$$

with $\mu_\Lambda(\rho)$ defined by (3.7). More generally, considering the same particle density ρ, one has exactly the same kinds of Bose condensation in the thermodynamic limit for the model SX in the *grand-canonical* ensemble as for the two models SX and X in the *canonical* ensemble. Therefore, the study of Bose condensation phenomena in the *grand-canonical* ensemble for the superstabilized model SX (1.18) also gives an analysis of the Bose condensation phenomena in the *canonical* ensemble for the model X (1.17).

In particular, assuming as verified all the assumptions of theorem 3.2, by corollary 4.2, i.e. (4.32), combined with corollary 3.5, a 'global' Bose condensation (3.37) for the model X (1.17) in the *grand-canonical* ensemble implies the existence of the same 'global' Bose condensation (3.37) for the model X (1.17) in the *canonical* ensemble.

Then, for $d \geqslant 3$, considering that the IBG (1.1) manifests, for high densities $\rho \geqslant \rho^{PBG}(\beta, 0)$ (1.9), only one conventional BE condensation $\rho_0^{IBG}(\beta, \rho)$ (1.12) in the *grand-canonical* ensemble, equation (4.32) implies the existence of only one conventional BE condensation for the IBG or the PBG (1.2) in the *canonical* ensemble with the same condensate density (cf [29]),

$$\begin{aligned}\lim_{\Lambda} \frac{1}{V} \langle a_0^* a_0\rangle_{H_\Lambda^{IBG}} \left(\beta, \mu_\Lambda^{IBG}(\rho)\right) &= \lim_{\Lambda} \frac{1}{V} \langle a_0^* a_0\rangle_{T_\Lambda} \left(\beta, \alpha_\Lambda^{PBG}(\rho)\right) \\ &= \lim_{\Lambda} \frac{1}{V} \langle a_0^* a_0\rangle_{H_\Lambda^{IBG}}(\beta, \rho) = \lim_{\Lambda} \frac{1}{V} \langle a_0^* a_0\rangle_{T_\Lambda}(\beta, \rho) \\ &= \sup\{0, \rho - \rho^{PBG}(\beta, 0)\}\end{aligned} \tag{4.33}$$

see (1.12)–(1.15), with $\alpha_\Lambda^{PBG}(\rho)$ and $\mu_\Lambda^{IBG}(\rho)$ respectively defined by (1.16) and (4.24). More generally, using conjecture 3.8 combined with theorem 4.1 and corollary 4.2 (see also (4.32)) we express our second conjecture.

Conjecture 4.4. *If some sufficient conditions such as conditions 2.1 are verified, for a fixed particle density* $\rho > 0$, *the model X (1.17) manifests in the canonical and grand-canonical ensembles exactly the same kinds of condensation, type included.*

5. Concluding remarks

By adding the interaction $(\lambda/V)N_\Lambda^2$, see (1.18), the paper [1] proposes a method to superstabilize a non-superstable Hamiltonian H_Λ^X (1.17) verifying conditions 2.1 and more precisely the *weak equivalence* of ensembles (2.10). This is done without destroying the 'fundamental' thermodynamic properties coming from the Bose system X (1.17), see [1] or section 2. In particular, in this paper we show that the same 'global' Bose condensation should appear in the two models X (1.17) and SX (1.18), see corollary 3.5. More generally, we conjecture that the Bose systems X and SX manifest exactly the same kinds of condensation, *type included*, see conjecture 3.8. An example is given by theorem 3.6 and corollary 3.7, or more precisely by the specific model (3.40), see [12, 13].

In fact, we finally explain that this procedure restores, in the thermodynamic limit, the *strong equivalence* of ensembles, cf theorem 4.1 and corollary 4.2, see also [2, 21–26]. A direct consequence of these close thermodynamic relations concerns the study of the original model X (1.17) in the *canonical* ensemble using the superstable gas SX (1.18). Thus this method of superstabilization (1.18) is also an analysis method of the model X in the *canonical* ensemble using its thermodynamic behaviour in the *grand-canonical* ensemble. As an application, for the model X, the Bose condensation mechanisms in the two ensembles, canonical and grand-canonical, are very close, see (4.32). In particular, using conjecture 3.8, these condensation phenomena seem to be identical (type included) in the canonical and grand-canonical ensembles, see conjecture 4.4.

Remark 5.1. Note that conditions 2.1 represent only some sufficient conditions. For example, assumption (iv) of conditions 2.1 could be relaxed. Indeed, we can generalize these previous results even if the (infinite volume) particle density $\rho^X(\beta, \alpha < \alpha_{\text{sup}})$ (2.11) is continuous except for a finite number $l \geqslant 1$ of chemical potential $\{\alpha_{1,\beta}, \ldots, \alpha_{l,\beta}\} \subset (-\infty, \alpha_{\text{sup}})$.

Remark 5.2. The results of this paper remain the same if, instead of $(\lambda/V)N_\Lambda^2$ in (1.18), we use the 'forward scattering' interaction

$$\frac{\lambda}{V} \sum_{k_1, k_2 \in \Lambda^*} a_{k_1}^* a_{k_2}^* a_{k_2} a_{k_1} \qquad \lambda > 0.$$

Applying this superstabilization (1.18) to the PBG (1.2), one gets the IBG (1.1). Then, using these results above, we find again the complete thermodynamic behaviour of the IBG which highlights the canonical and grand-canonical thermodynamic relations between the PBG and the IBG, cf (3.39), (4.22)–(4.30) and (4.33), see also [1, 4–9, 14, 28–32].

To conclude, the status of the Bogoliubov approximation

$$a_0^\# / \sqrt{V} \to c^\# \in \mathbb{C} \qquad a_0^\# = \{a_0 \text{ or } a_0^*\} \qquad c^\# = \{c \text{ or } \overline{c}\}$$

should be analysed for the corresponding *non-superstable* model X (1.17). Indeed, on the one hand, this Bogoliubov approximation is already verified for any *superstable* model SX [34]. On the other hand, their thermodynamic behaviour is similar, see section 3.

Another question for the superstabilized model SX concerns the analysis of the exactness of the large deviation principle applied to a 'semi-local' density of particles in a subdomain $\widetilde{\Lambda}$ as is done for the PBG, see [35]. However all the corresponding explanations, including other analyses of some interacting Bose systems, different from the PBG and the IBG, should be reserved for following papers as some applications of those general results.

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Note added in proof. This superstabilization method was already used in 1992 [48], see also discussions in [19, 49–50]. Actually, the microscopic (Bogoliubov) theory of superfluidity involves a positive chemical potential to obtain the Landau (gapless) spectrum where the system corresponding to the Bogoliubov Hamiltonian is unstable [48]. Then, in [48–50] the authors use the forward scattering interaction to superstabilize the Bogoliubov model and find a gapless spectrum by different "Bogoliubov approximations".

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Appendix A. Classification of Bose condensations

A.1. The van den Berg–Lewis–Pulè classification of conventional generalized condensations

For reader's convenience, we recall here a nomenclature of (generalized) Bose–Einstein (BE) condensations according to [36–38]:

- the condensation is called *type I* when a *finite* number of single-particle levels are macroscopically occupied (e.g. the standard BE condensation in the one-mode for the PBG (1.2) is of type I);
- it is of *type II* when an *infinite* number of the levels are macroscopically occupied;
- it is called *type III*, or the *non-extensive* condensation, when *none* of the levels are macroscopically occupied whereas one has

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,0\leqslant\|k\|<\delta\}}\langle N_k\rangle = \sup\{0,\rho-\rho_c(\beta)\}\qquad \text{with}\qquad N_k\equiv a_k^*a_k.$$

An example of these different condensations is given in [36]. This paper demonstrates that three types of BE condensation can be realized in the case of the PBG (1.2) in an *anisotropic* rectangular box $\Lambda\subset\mathbb{R}^3$ of volume $V=|\Lambda|=L_1L_2L_3$ and with the Dirichlet boundary conditions. Let $L_1=V^{n_1}$, $L_2=V^{n_2}$, $L_3=V^{n_3}$ for $n_1+n_2+n_3=1$ and $n_1\geqslant n_2\geqslant n_3>0$. If $n_1<\frac{1}{2}$, then for sufficiently large density ρ, we have the BE condensation of *type I* in the fundamental mode $k=\left(\frac{2\pi}{L_1},\frac{2\pi}{L_2},\frac{2\pi}{L_3}\right)$. For $n_1=\frac{1}{2}$ one gets a condensation of *type II* characterized by a macroscopic occupation of infinite package of modes $k=\left(\frac{2\pi n}{L_1},\frac{2\pi}{L_2},\frac{2\pi}{L_3}\right)$, $n\in\mathbb{N}$, whereas for $n_1>\frac{1}{2}$ we obtain a condensation of *type III*. In [39, 40] it was shown that type III condensation can be caused in the PBG by a weak external potential or (see [37, 41]) by a specific choice of boundary conditions and geometry. Another example of the *non-extensive* condensation is given in an *isotropic* box Λ by the IBG (1.1) with the repulsive interaction

$$\frac{g}{V}\sum_{k\in\Lambda^*}a_k^*a_k^*a_ka_k\qquad g>0$$

which spreads out the *conventional* BE condensation of type I into the type III BE condensation, see [12, 42].

A.2. Non-conventional and conventional Bose condensations

The Bose condensations could also be classified by their mechanisms of formation, see [10]. In the overwhelming majority of papers (cf [36, 37, 39–42]), the condensation is due to *saturation* of the particle density, originally discovered by Einstein [43] in the Bose gas without interaction (PBG).

That kind of Bose condensation is called a *conventional* BE condensation [10, 14].

The existence of condensations, which is induced by *interaction*, is pointed out in recent papers [10, 12, 13, 19, 20, 44]. It is also the case of the Huang–Yang–Luttinger or the full diagonal models, since the Bose condensation manifested by the two models exists in the presence of *attractive* interactions, see [45–47]. In particular note that in the Huang–Yang–Luttinger model, the condensation appears via a first-order phase transition [46, 47]. This is also the case for the Bogoliubov weakly imperfect Bose gas (cf equation (3.81) in [18] and [10, 19, 20]).

The kind of condensation induced by the attraction mechanism is denoted as *non-conventional* Bose condensation, cf [10].

Remark A.1. A non-conventional Bose condensation can always be characterized by its type. Therefore, formally one obtains six kinds of condensation: non-conventional/conventional of types I, II or III.

Appendix B. Gibbs states and generating functional

The purpose of this section is to review the characterization of (Gibbs) states of a Bose system by their generating functional, a method originally introduced by Araki and Woods in the case of the PBG [28]. For each Gibbs state, there is a representation of the canonical commutation relations (CCR) given by the GNS construction. For a complete description see [32], and also [29–31] for a detailed analysis of the PBG Gibbs state. Here, we only present a quick overview.

Let M be a complex pre-Hilbert space with the corresponding scalar product $(.,.)_M$. We consider a representation of the CCR over M given by a map $h \mapsto W(h)$ from M to a space $U(\mathcal{H})$ of unitary operators on a Hilbert space $\mathcal{H}$ satisfying

$$W(h_1)W(h_2) = \exp\left\{-\frac{\mathrm{i}}{2}\,\mathrm{Im}\,(h_1,h_2)_M\right\} W(h_1+h_2) \tag{B.1}$$

and such that the map $\lambda \mapsto W(\lambda h)$ from $\mathbb{R}$ to $U(\mathcal{H})$ is strongly continuous. By Stone's theorem [32], the continuity implies the existence of self-adjoint operators $R(h)$ such that

$$W(h) = \mathrm{e}^{\mathrm{i}R(h)}. \tag{B.2}$$

The $R(h)$ are called the *field operators* and can be interpreted as the random variables of a non-commutative probability theory, since by (B.1) one gets

$$[R(h_1), R(h_2)] = \mathrm{i}\,\mathrm{Im}(h_1,h_2)_M. \tag{B.3}$$

Note that the map $h \to R(h)$ is a linear over $\mathbb{R}$, but anti-linear over $\mathrm{i}\mathbb{R}$. For $h \in M$, we can now define the *creation* and *annihilation* operators $a^*(h)$ and $a(h) \equiv (a^*(h))^*$ by

$$a^*(h) \equiv \frac{1}{\sqrt{2}}\{R(h) - \mathrm{i}R\,(\mathrm{i}h)\} \qquad a(h) \equiv \frac{1}{\sqrt{2}}\{R(h) + \mathrm{i}R(\mathrm{i}h)\}. \tag{B.4}$$

A representation of the CCR is called *cyclic* if there is a vector Ω in $\mathcal{H}$ such that the set $\{W(h)\Omega\}_{h\in M}$ is dense in $\mathcal{H}$. Such Ω is called a cyclic vector. It can be shown that, for every *regular* Gibbs state $\langle\cdot\rangle$, there is a unique (up to unitary equivalence) representation of the CCR with cyclic vector Ω such that

$$\langle \mathrm{e}^{\mathrm{i}R(h)}\rangle = (\Omega, W(h)\Omega)_{\mathcal{H}}.$$

The *generating functional* of the representation is defined by

$$\mathbb{E}(h) \equiv (\Omega, W(h)\Omega)_{\mathcal{H}} \qquad h \in M. \tag{B.5}$$

The generating functional plays the same role for a state on the CCR algebra as the characteristic function for probability distribution, see [32].

Theorem B.1. *(Araki–Segal). Let $\mathbb{E}$ be the generating functional of a cyclic representation of the CCR over M. Then it satisfies the following:*

(i) $\mathbb{E}(0) = 1$;

(ii) for any finite set $\{c_j \in \mathbb{C}; h_j \in M\}$, *one has*

$$\sum_{l,s=1}^{n} \mathbb{E}(h_l - h_s) \exp\left\{\frac{\mathrm{i}}{2} \operatorname{Im}(h_l, h_s)_M\right\} \overline{c_l} c_s \geqslant 0;$$

(iii) for $h \in M$, *the map* $\lambda \to \mathbb{E}(\lambda h)$ *from* $\mathbb{C}$ *to* $\mathbb{R}$ *is continuous.*

Conversely, any generating functional $\mathbb{E} : M \to \mathbb{C}$ *satisfying (i), (ii) and (iii) is a generating functional of a cyclic representation of the CCR.*

Our concrete set-up will be as follows. For a (sufficiently regular) *finite* volume, $\Lambda \subset \mathbb{R}^d$, the grand-canonical Gibbs state $\langle\cdot\rangle_\Lambda(\beta, \mu)$, is defined on the set of bounded operators acting on the boson Fock space $\mathcal{F}^B_\Lambda \equiv \mathcal{F}^B(L^2(\Lambda))$ over $L^2(\Lambda)$, see (1.4). In order to analyse the state $\langle\cdot\rangle_\Lambda(\beta, \mu)$, we use the Fock representation $W^{\mathcal{F}^B_\Lambda}$ of the CCR over the pre-Hilbert space $M = \mathcal{D}_\Lambda$ (the space of the $C_0^\infty(\Lambda)$-functions with compact support contained in Λ). Its generating functional (B.5) is equal to $\mathbb{E}_{\mathcal{F}^B_\Lambda}(h) = \mathrm{e}^{-\frac{1}{4}\|h\|^2}$, where cyclic vector Ω is vacuum in $\mathcal{H} = \mathcal{F}^B_\Lambda$: $a(h)\Omega = 0$ for any $h \in \mathcal{D}_\Lambda$. Since $\mathcal{D}_\Lambda$ is dense in $L^2(\Lambda)$, one can extend $W^{\mathcal{F}^B_\Lambda}$ to the latter. We shall calculate the generating functional

$$\mathbb{E}_\Lambda(\beta, \mu; h) \equiv \left\langle W^{\mathcal{F}^B_\Lambda}(h)\right\rangle_\Lambda(\beta, \mu) \qquad h \in \mathcal{D}_\Lambda \tag{B.6}$$

and study its thermodynamic limit ($\Lambda \uparrow \mathbb{R}^d$).

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